$HCR's\ Formula\ For\ Regular\ \ Spherical\ Polygon$

It is a very important formula (mathematical relation) applicable on any regular spherical polygon having each of its sides as an arc of the great circle on a spherical surface. This formula has already been derived by the author H.C. Rajpoot in his paper "**Mathematical Analysis of Regular Spherical Polygons**". Here is another derivation of this formula using HCR's cosine formula.

For any regular spherical polygon, having n number of sides each as a great circle arc of length a & each interior angle θ, drawn on a spherical surface with a radius R then all these four parameters are related by HCR's formula for regular spherical polygon as follows

$$cos\left(\frac{a}{2R}\right)sin\left(\frac{\theta}{2}\right)sec\left(\frac{\pi}{n}\right) = 1$$

Derivation of formula for regular spherical polygon using HCR's cosine formula: Consider a regular spherical polygon, having $\boldsymbol{n}$ number of equal sides each as a great circle arc of length $\boldsymbol{a}$ & each interior angle $\boldsymbol{\theta}$, on a spherical surface of radius $\boldsymbol{R}$. Now, the angle subtended by each side as a great circle arc at the centre O of the sphere

$$\boldsymbol{\delta} = \frac{\text{Arc length of side}}{\text{radius of sphere}} = \frac{\boldsymbol{a}}{\boldsymbol{R}}$$

If we join all the vertices of the regular spherical polygon, having $\boldsymbol{n}$ number of sides each as a great circle arc, by the straight lines through the interior of sphere then we get a regular plane polygon having $\boldsymbol{n}$ number of sides then each interior angle (between any two adjacent sides) of regular plane polygon

$$\boldsymbol{\theta_p} = \frac{\text{total sum of interior angles}}{\text{number of sides}} = \frac{(\boldsymbol{n}-\boldsymbol{2})\boldsymbol{\pi}}{\boldsymbol{n}}$$

Now, consider any two adjacent sides as the great circle arcs AB & AC of equal length $\boldsymbol{a}$ subtending equal angle $\boldsymbol{\delta}$ at the centre O of the sphere & meeting each other at angle $\boldsymbol{\theta}$ at any vertex A of the regular spherical polygon. Join the end points A, B & C of the great circle arcs AB & AC by the dotted straight lines through the interior of sphere to obtain the chords AB & AC meeting at the same vertex A (As shown in the figure 1) then the (plane) angle between the chords AB & AC of the great circle arcs AB & AC is given by cosine formula

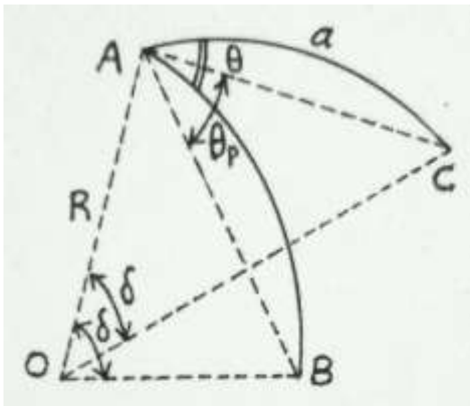

Figure 1: Two great circle arcs AB & AC of equal length $\boldsymbol{a}$ meeting each other at angle $\boldsymbol{\theta}$ at vertex A of regular spherical polygon & subtending equal angle $\boldsymbol{\delta}$ at the centre O of sphere of radius R. The chords AB & AC making angle $\boldsymbol{\theta_p}$ are drawn through the interior of sphere

$$cos\theta_p = sin\frac{\alpha}{2}sin\frac{\beta}{2} + cos\frac{\alpha}{2}cos\frac{\beta}{2}cos\theta$$

$$\text{Now, setting the corresponding values, } \theta_p = \frac{(n-2)\pi}{n}\ ,$$

$$\alpha = \beta = \delta = \frac{a}{R}\ \text{(since, the arcs AB \& } AC \text{ are of equal length } a\text{), we get}$$

$$cos\frac{(n-2)\pi}{n} = sin\frac{a}{2R}sin\frac{a}{2R} + cos\frac{a}{2R}cos\frac{a}{2R}cos\theta$$

$$cos\left(\pi - \frac{2\pi}{n}\right) = sin^2\frac{a}{2R} + cos^2\frac{a}{2R}cos\theta$$

$$-cos\frac{2\pi}{n} = sin^2\frac{a}{2R} + cos^2\frac{a}{2R}\left(1 - 2sin^2\frac{\theta}{2}\right)$$

$$-\left(2cos^2\frac{\pi}{n}-1\right)=sin^2\frac{a}{2R}+cos^2\frac{a}{2R}-2sin^2\frac{\theta}{2}cos^2\frac{a}{2R}$$

$$1-2cos^2\frac{\pi}{n}=1-2sin^2\frac{\theta}{2}cos^2\frac{a}{2R}$$

$$cos^2\frac{\pi}{n}=sin^2\frac{\theta}{2}cos^2\frac{a}{2R}$$

Taking square roots on both the sides, we get

$$\left|\cos\frac{\pi}{n}\right|=\left|sin\frac{\theta}{2}cos\frac{a}{2R}\right|$$

since, $\mathbf{n\geq 3}$, $\boldsymbol{\frac{(n-2)\pi}{n}<\theta<\pi}$ & $\boldsymbol{a<\frac{2\pi R}{n}}$ hence we get

$$cos\frac{\pi}{n}=sin\frac{\theta}{2}cos\frac{a}{2R}$$

$$\boldsymbol{sin\frac{\theta}{2}cos\frac{a}{2R}sec\frac{\pi}{n}=1}$$

Above is HCR's formula for regular spherical polygon. It is very important formula which co-relates four important parameters i.e. number of sides $\boldsymbol{n}$, interior angle $\boldsymbol{\theta}$, arc length of side $\boldsymbol{a}$ & radius of sphere $\boldsymbol{R}$ in any regular spherical polygon. Above formula is of crucial importance to find out any of the four important parameters R, a, n & θ if other three are given (known) in any regular spherical polygon while n is always a positive integer $(n\geq 3)$. It also concludes that any three of the four parameters R, a, n & θ are self-sufficient to exactly represent a unique regular spherical polygon because if three parameters are known then the forth unknown is computed by the above formula.

It is to be noted that a regular spherical polygon having three known parameters $\boldsymbol{a}, \boldsymbol{n}$ & $\boldsymbol{\theta}$ can be created or drawn only on a unique spherical surface of a radius R which is given by HCR's formula as

$$R=\frac{a}{2\cos^{-1}\left(cos\left(\frac{\pi}{n}\right)cosec\left(\frac{\theta}{2}\right)\right)}$$

$\boldsymbol{R}=radius\ of\ the\ spherical\ surface$

$\boldsymbol{n}=no.of\ sides\ of\ regular\ spherical\ polygon$ $\quad(\forall\ n\in N\ \&\ n\geq 3)$

$\boldsymbol{a}=length\ of\ each\ side\ of\ regular\ spherical\ polygon$ $\quad\left(\forall\ a<\frac{2\pi R}{n}\right)$

$\boldsymbol{\theta}=\ interior\ angle\ of\ \ regular\ spherical\ polygon$ $\quad\left(\forall\ \frac{(n-2)\pi}{n}<\theta<\pi\right)$