"HCR's Theory of Polygon"

"Solid angle subtended by any polygonal plane at any point in 3D space" 9 Oct, 2014

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Proposed & illustrated by H. C. Rajpoot

I. Introduction

The graphical method overcomes the limitations of all other analytical methods provided the location of foot of perpendicular drawn from the given point to the plane of polygon is known. It involves theoretically zero error if the calculations are done correctly. A polygon is the plane bounded by the straight lines. But, if the given polygon is divided into certain number of elementary triangles then the solid angle subtended by polygon at given point can be determined by summing up the solid angles subtended by all the elementary triangles at the same point. This concept derived a Theory. According to it "for a given configuration of plane & location of point in the space, if a perpendicular, from any given point in the space, is drawn to the plane of given polygon then the polygonal plane (polygon) can be internally or externally or both divided into a certain number of elementary triangles by joining all the vertices of polygon to the foot of perpendicular (**F.O.P.**) Further each of these triangles can be internally or externally sub-divided in two right triangles having common vertex at the foot of perpendicular. Thus the solid angle subtended by the given polygonal plane at the given point is the algebraic sum of solid angles subtended by all the elementary triangles at the same point such that algebraic sum of the areas of all these triangles is equal to the area of given polygonal plane"

Let's study in an order to easily understand the Theory of Polygon in a simple way

II. HCR's Master/Standard Formula-1 (Solid angle subtended by a right triangular plane at any point lying at a normal height h from any of the acute angled vertices)

Using Fundamental Theorem of Solid Angle:

Let there be a right triangular plane ONM having perpendicular ON = p & the base MN = b and a given point say P(0, 0, h) at a height 'h' lying on the axis (i.e. Z-axis) passing through the acute angled vertex say 'O'

(As shown in the figure1 below)

In rightΔPOR

$$\Rightarrow sec\varphi = \frac{PR}{PO} \quad \Rightarrow PR = POsec\varphi = hsec\varphi \ \ \&$$

$$tan\varphi = \frac{OR}{PO} \quad \Rightarrow OR = POtan\varphi = htan\varphi = x$$

Now, the equation of the straight line OM passing through the origin 'O'

$$\Rightarrow y = mx = \frac{b}{p}x \quad \left(since, \qquad slope\ of\ line\ OM = \frac{b}{p}\right)$$

$$\Rightarrow y = \frac{b}{p}x = \frac{b}{p}htan\varphi$$

Now, consider a point Q(x, y) on the straight line OM

$$\Rightarrow QR = y = \frac{b}{p} h tan\varphi$$

In rightΔPRQ

$$\Rightarrow sin\theta = \frac{QR}{PQ} = \frac{QR}{\sqrt{QR^2 + PR^2}} = \frac{\left(\frac{b}{p} h tan\varphi\right)}{\sqrt{\left(\frac{b}{p} h tan\varphi\right)^2 + h^2 sec^2\varphi}}$$

$$sin\theta = \frac{bsin\varphi}{\sqrt{b^2 sin^2\varphi + p^2}}$$

In rightΔPON

$$\Rightarrow cos\varphi_0 = \frac{OP}{PN} = \frac{OP}{\sqrt{PO^2 + ON^2}} = \frac{h}{\sqrt{h^2 + p^2}}$$

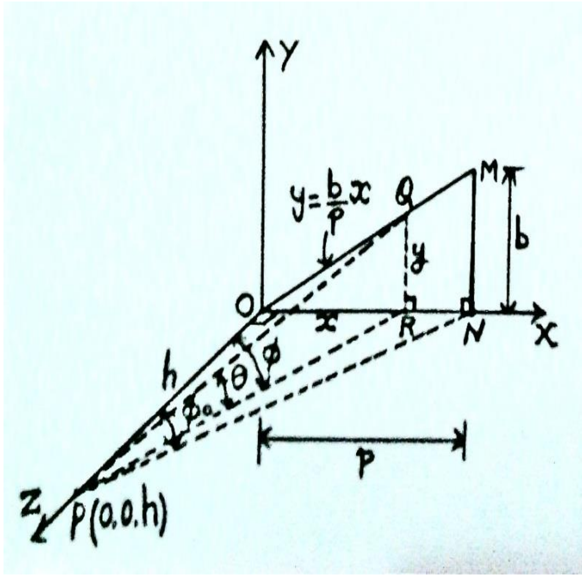


Fig 1: Right Triangular Plane ONM, ON=p & MN=b

Let's consider an imaginary spherical surface having radius 'R' & centre at the given point 'P' such that the area of projection of the given plane ONM on the spherical surface w.r.t. the given point 'P' is 'A'

Now, consider an elementary area of projection 'dA' of the plane ONM on the spherical surface in the first quadrant YOX

(As shown in the figure 2)

Now, elementary area of projection in the first quadrant

$$\Rightarrow dA = (length)(width) = (Rsin\theta d\varphi)(Rd\theta) = R^2 sin\theta d\theta d\varphi$$

Hence, area of projection of the plane ONM on the spherical surface in the first quadrant is obtained by integrating the above expression in the first quadrant &

Applying the proper limits of $'\theta' from\ \theta_1 = (\pi/2 - \theta)\ to\ \theta_2 = \pi/2$ & $'\varphi' from\ \varphi_1 = 0\ to\ \varphi_2 = \varphi_0,\ we\ get$

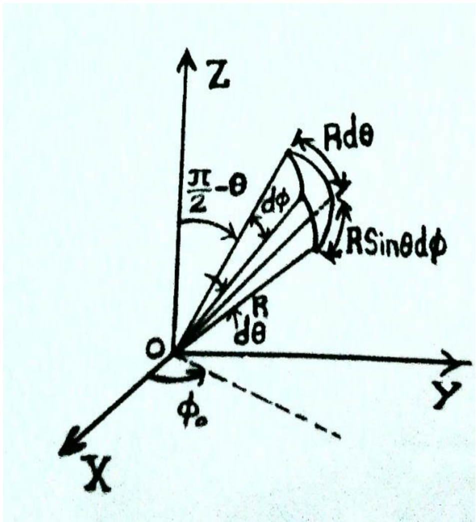


Fig 2: Elementary area dA on spherical surface

$$\Rightarrow A = \int_0^{\varphi_0} \left[\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} R^2 sin\theta d\theta\right] d\varphi = R^2 \int_0^{\varphi_0} \left[\int_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} sin\theta d\theta\right] d\varphi$$

$$= R^2 \int_0^{\varphi_0} d\varphi\, [-cos\theta]_{\frac{\pi}{2}-\theta}^{\frac{\pi}{2}} = R^2 \int_0^{\varphi_0} sin\theta d\varphi = R^2 \int_0^{\varphi_0} \frac{bsin\varphi}{\sqrt{b^2 sin^2\varphi + p^2}} d\varphi$$

$$= R^2 \int_0^{\varphi_0} \frac{bsin\varphi}{\sqrt{b^2 - b^2 sin^2\varphi + p^2}} d\varphi \qquad (since,\ sin^2\alpha = 1 - cos^2\alpha)$$

$$= R^2 \int_0^{\varphi_0} \frac{bsin\varphi}{b\sqrt{\frac{b^2+p^2}{b^2} - cos^2\varphi}} d\varphi = R^2 \int_0^{\varphi_0} \frac{sin\varphi}{\sqrt{K^2 - cos^2\varphi}} d\varphi$$

$$where, K = \sqrt{\frac{b^2+p^2}{b^2}} = \frac{\sqrt{b^2+p^2}}{b}$$

$$\Rightarrow A = R^2 \left[-\sin^{-1}\left\{\frac{cos\varphi}{K}\right\}\right]_0^{\varphi_0} = R^2 \left[-\sin^{-1}\left\{\frac{cos\varphi_0}{K}\right\} + \sin^{-1}\left\{\frac{cos0}{K}\right\}\right]$$

$$= R^2 \left[\sin^{-1}\left\{\frac{1}{K}\right\} - \sin^{-1}\left\{\frac{cos\varphi_0}{K}\right\}\right]$$

Now, on setting the values of 'K' & $'cos\varphi_0'$, we get

$$\Rightarrow A = R^2 \left[sin^{-1}\left\{\frac{1}{\frac{\sqrt{b^2+p^2}}{b}}\right\} - sin^{-1}\left\{\frac{\frac{h}{\sqrt{h^2+p^2}}}{\frac{\sqrt{b^2+p^2}}{b}}\right\}\right]$$

$$= R^2 \left[sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}\right]$$

Now, the solid angle subtended by the right triangular plane ONM at the given point 'P'

= solid angle subtended by area of projection of triangle ONM on the spherical surface at the same point 'P'

Using Fundamental Theorem

$$\Rightarrow \omega = \int_s \frac{dA}{r^2} = \int_s \frac{dA}{R^2} = \frac{1}{R^2}\int_s dA = \frac{A}{R^2}$$

$(since,\ R\ is\ constant\ for\ each\ point\ on\ the\ area\ of\ projection\ on\ the\ spherical\ surface)$

$$\Rightarrow \omega = \frac{1}{R^2} \times R^2 \left[\sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}\right]$$

$$= \sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

$$\Rightarrow \boldsymbol{\omega} = \left[\mathbf{sin^{-1}}\left\{\frac{\boldsymbol{b}}{\sqrt{\boldsymbol{b^2+p^2}}}\right\} - \mathbf{sin^{-1}}\left\{\left(\frac{\boldsymbol{b}}{\sqrt{\boldsymbol{b^2+p^2}}}\right)\left(\frac{\boldsymbol{h}}{\sqrt{\boldsymbol{h^2+p^2}}}\right)\right\}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

If the given point P (0, 0, h) is lying on the axis normal to the plane & passing through the acute angled vertex 'M' then the solid angle subtended by the right triangular plane ONM is obtained by replacing 'b' by 'p' & 'p' by 'b' in the equation(1), we have

$$\Rightarrow \omega = \left[\sin^{-1}\left\{\frac{p}{\sqrt{p^2+b^2}}\right\} - \sin^{-1}\left\{\left(\frac{p}{\sqrt{p^2+b^2}}\right)\left(\frac{h}{\sqrt{h^2+b^2}}\right)\right\}\right]$$

Note: Eq(1) is named as **HCR's Master/Standard Formula-1** which is **extremely useful** to find out the solid angle subtended by any polygon at any point in the space thus all the formulae can be derived by using eq(1).

III. Specifying the location of given point & foot of perpendicular in the Plane of Polygon

Let there be any point say point P & any polygonal plane say plane ABCDEF in the space. Draw a perpendicular PO from the point 'P' to the plane of polygon which passes through the point 'O' i.e. **foot of perpendicular (F.O.P.)** (See front & top views in the figure 3 below showing actual location of point 'P' & F.O.P. 'O')

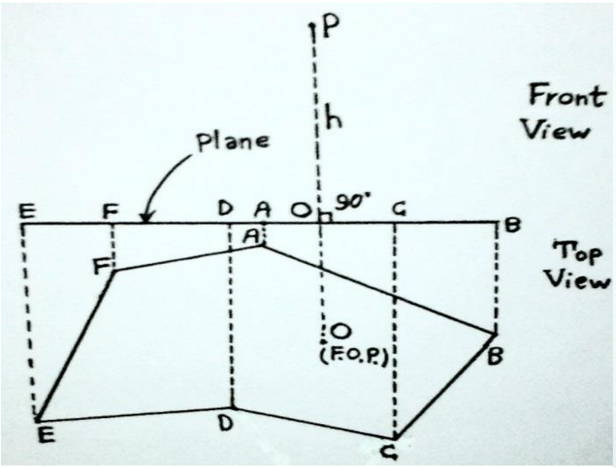


Fig 3: **Actual location of point 'P' & F.O.P. 'O'**

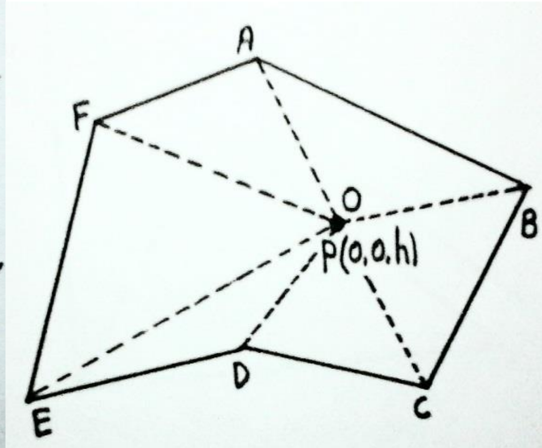


Fig 4: **Location of 'P' & 'O' in the plane of paper**

Now, for simplification of specifying the location of given point 'P' & foot of perpendicular 'Q' we assume

a. **Foot of perpendicular 'O' as the origin &**
b. **Point P is lying on the Z-axis**

If the length of perpendicular PO is h then the location of given point 'P' & foot of perpendicular 'O' in the plane of polygon (i.e. plane of paper) is denoted by $P(0,0,h)$ similar to the 3-D co-ordinate system

(See the figure 4 above)

IV. Element Method (Method of dividing the polygon into elementary triangles)

It is the method of Joining all the vertices A, B, C, D, E & F of a given polygon to the foot of perpendicular 'O' drawn from the given point P in the space (as shown by the dotted lines in figure 4 above) Thus, $\Delta \boldsymbol{AOB}, \Delta \boldsymbol{BOC}, \Delta \boldsymbol{COD}, \Delta \boldsymbol{DOE}, \Delta \boldsymbol{EOF}$ & $\Delta \boldsymbol{AOF}$ are the **elementary triangles** which have common vertex at the foot of perpendicular 'O'.

Note: $\Delta ABC, \Delta BCD, \Delta CDE, \Delta DEF$ & ΔEFA are not taken as the elementary triangles since they don't have common vertex at the foot of perpendicular 'O'

The area $(A_{polygon})$ of polygon ABCDEF is given as follows (from the figure 4)

$$A_{polygon} = algebraic\ sum\ of\ areas\ of\ elementary\ triangles$$
$$= A_{\Delta AOB} + A_{\Delta BOC} + A_{\Delta COD} + A_{\Delta DOE} + A_{\Delta EOF} + A_{\Delta AOF}$$

Since the **location of given point & the configuration of polygonal plane is not changed** hence the solid angle $(\omega_{polygon})$ subtended by polygonal plane at the given point in the space is the algebraic sum of solid angles subtended by the elementary triangles obtained by joining all the vertices of polygon to the F.O.P.

Now, replacing the areas of elementary triangles by their respective values of solid angles in the above expression as follows

$$\omega_{polygon} = alebraic\ sum\ of\ solid\ angles\ subtended\ by\ the\ elementary\ triangles$$

$$= \omega_{\Delta AOB} + \omega_{\Delta BOC} + \omega_{\Delta COD} + \omega_{\Delta DOE} + \omega_{\Delta EOF} + \omega_{\Delta AOF}$$

While, the values of solid angles subtended by elementary triangles are determined by using **axiom of triangle** i.e. dividing each elementary triangle into two right triangles & using standard formula-1 of right triangle.

V. Axiom of Triangle

"If the perpendicular, drawn from a given point in the space to the plane of a given triangle, passes through one of the vertices then that triangle can be divided internally or externally (w.r.t. F.O.P.) into two right triangles having common vertex at the foot of perpendicular, simply by drawing a normal from the common vertex (i.e. F.O.P.) to the opposite side of given triangle."

- An **acute angled triangle** is **internally divided** w.r.t. F.O.P. (i.e. common vertex)
- A **right angled triangle** is **internally divided** w.r.t. F.O.P. (i.e. common vertex)
- An **obtuse angled triangle** is divided

Internally if and only if the **angle of common vertex (F.O.P.) is obtuse**

Externally if and only if the **angle of common vertex (F.O.P.) is acute**

Now, consider a given point $P(0,0,h)$ (located **perpendicular to the plane of paper**) lying at a height h on the normal axes passing through the vertices A, B & C where the points $P_1(0,0,h), P_2(0,0,h)$ & $P_3(0,0,h)$ are the different locations of given point P on the perpendiculars passing through the vertices A, B & C respectively at the same height h.

(See the different cases in the figures (5), (6) & (7) below)

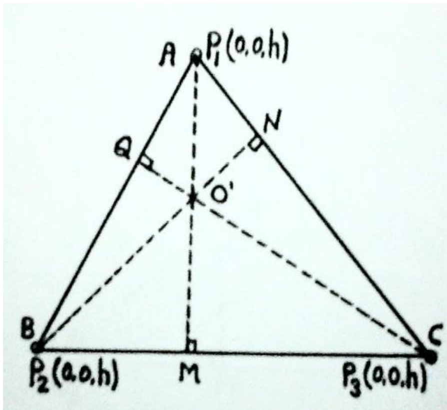


Fig 5: Acute Angled Triangle

- **Acute Angled Triangle:**

Consider the given point $\boldsymbol{P_1(0,0,h)}$ at a normal height h from the vertex 'A' (i.e. foot of perpendicular drawn from the point P_1 to the plane ofΔABC)

Now, draw the perpendicular AM from F.O.P. 'A' to the opposite side BC to divide the ΔABC into two sub-elementary right triangles ΔAMB & ΔAMC

In this case, the area $(A_{\Delta ABC})$ of ΔABC is given by

$$A_{\Delta ABC} = algebraic\ sum\ of\ areas\ of\ elemetary\ triangles$$

$$= A_{\Delta AMB} + A_{\Delta AMC} \qquad (\boldsymbol{\Delta ABC\ is\ internally\ divided})$$

Hence replacing the areas by the corresponding values of solid angles, we get

$$\omega_{\Delta ABC} = algebraic\ sum\ of\ solid\ angles\ subtended\ by\ elemetary\ triangles\ at\ point\ P_1(0,0,h)$$

$$= \omega_{\Delta AMB} + \omega_{\Delta AMC} \qquad (\boldsymbol{\Delta ABC\ is\ internally\ divided})$$

The values of $\omega_{\Delta AMB}$ & $\omega_{\Delta AMC}$ subtended by the right triangles ΔAMB & ΔAMC respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

Measure the dimensions AM, BM & P_1A from the drawing & set $b = BM, p = AM$ &

$h = normal\ height\ of\ point\ P_1(0,0,h) from\ the\ vertex\ 'A'(i.e. F.O.P.) = P_1A$ in above expression

We get, the solid angle subtended by the right ΔAMB at the given point $P_1(0,0,h)$

$$\omega_{\Delta AMB} = \sin^{-1}\left\{\frac{BM}{\sqrt{(BM)^2 + (AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BM}{\sqrt{(BM)^2 + (AM)^2}}\right)\left(\frac{P_1A}{\sqrt{(P_1A)^2 + (AM)^2}}\right)\right\}$$

Similarly, solid angle subtended by the right ΔAMC at the the same point $P_1(0,0,h)$

$$\omega_{\Delta AMC} = \sin^{-1}\left\{\frac{CM}{\sqrt{(CM)^2 + (AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CM}{\sqrt{(CM)^2 + (AM)^2}}\right)\left(\frac{P_1A}{\sqrt{(P_1A)^2 + (AM)^2}}\right)\right\}$$

Hence, the solid angle subtended by the given ΔABC at the the given point $\boldsymbol{P_1(0,0,h)}$ is calculated as

$$\Rightarrow \ \boldsymbol{\omega_{\Delta ABC} = \omega_{\Delta AMB} + \omega_{\Delta AMC}}$$

Similarly, for the location of given point $\boldsymbol{P_2(0,0,h)}$ lying at a normal height h from the vertex 'B' (i.e. foot of perpendicular drawn from the point $P_2(0,0,h)$ to the plane of ΔABC) (See figure 5 above)

By following the above procedure, we get the solid angle subtended by the given ΔABC at the the given point $P_2(0,0,h)$ as follows

$$\Rightarrow \ \boldsymbol{\omega_{\Delta ABC} = \omega_{\Delta ANB} + \omega_{\Delta CNB}}$$

$$\omega_{\Delta ANB} = \sin^{-1}\left\{\frac{AN}{\sqrt{(AN)^2 + (BN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AN}{\sqrt{(AN)^2 + (BN)^2}}\right)\left(\frac{P_2B}{\sqrt{(P_2B)^2 + (BN)^2}}\right)\right\} \ \&$$

$$\omega_{\Delta CNB} = \sin^{-1}\left\{\frac{CN}{\sqrt{(CN)^2 + (BN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CN}{\sqrt{(CN)^2 + (BN)^2}}\right)\left(\frac{P_2B}{\sqrt{(P_2B)^2 + (BN)^2}}\right)\right\}$$

Similarly, for the location of given point $\boldsymbol{P_3(0,0,h)}$ lying at a normal height h from the vertex 'C' (i.e. foot of perpendicular drawn from the point $P_3(0,0,h)$ to the plane of ΔABC) (See figure 5 above)

By following the above procedure, we get the solid angle subtended by the given ΔABC at the the given point $P_3(0,0,h)$ as follows

$$\Rightarrow \ \boldsymbol{\omega_{\Delta ABC} = \omega_{\Delta AQC} + \omega_{\Delta BQC}}$$

We can calculate the corresponding values of $\omega_{\Delta AQC}$ & $\omega_{\Delta BQC}$ using standard formula-1 as follows

$$\omega_{\Delta AQC} = \sin^{-1}\left\{\frac{AQ}{\sqrt{(AQ)^2 + (CQ)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AQ}{\sqrt{(AQ)^2 + (CQ)^2}}\right)\left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CQ)^2}}\right)\right\} \ \&$$

$$\omega_{\Delta BQC} = \sin^{-1}\left\{\frac{BQ}{\sqrt{(BQ)^2 + (CQ)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BQ}{\sqrt{(BQ)^2 + (CQ)^2}}\right)\left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CQ)^2}}\right)\right\}$$

- **Right Angled Triangle:**

Consider the given point $\boldsymbol{P_1(0,0,h)}$ at a normal height h from the vertex 'A' (i.e. foot of perpendicular drawn from the point $P_1(0,0,h)$ to the plane of right ΔBAC) (See the figure 6)

Now, draw the perpendicular AM from F.O.P. 'A' to the opposite side BC to divide the right ΔBAC into two sub-elementary right triangles ΔAMB & ΔAMC

In this case, the area $(A_{\Delta BAC})$ of right ΔBAC is given by

$A_{\Delta BAC} = algebraic\ sum\ of\ areas\ of\ elemetary\ triangles$

$A_{\Delta BAC} =$
$A_{\Delta AMB} + A_{\Delta AMC}$ $\quad (\boldsymbol{right\ \Delta BAC\ is\ internally\ divided})$

Hence replacing the areas by the corresponding values of solid angles, we get

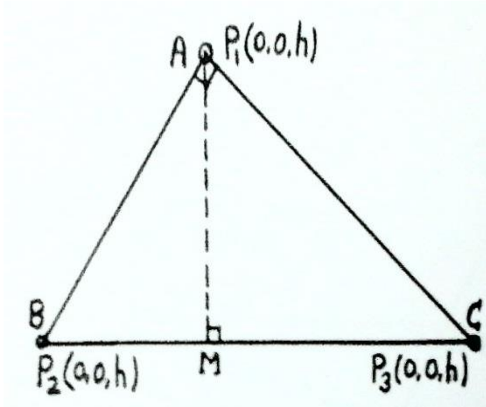


Fig 6: Right Angled Triangle

$$\omega_{\Delta BAC} = algebraic\ sum\ of\ solid\ angles\ subtended\ by\ elemetary\ triangles\ at\ point\ P_1(0,0,h)$$

$$= \omega_{\Delta AMB} + \omega_{\Delta AMC} \qquad (\boldsymbol{right\ \Delta BAC\ is\ internally\ divided})$$

The values of $\omega_{\Delta AMB}$ & $\omega_{\Delta AMC}$ subtended by the right triangles ΔAMB & ΔAMC respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

Measure the dimensions AM, BM & P_1A from the drawing & set $b = BM, p = AM$ &

$h = normal\ height\ of\ point\ P_1(0,0,h) from\ the\ vertex\ 'A'(i.e. F.O.P.) = P_1A$ in above expression

We get, the solid angle subtended by the right ΔAMB at the given point $P_1(0,0,h)$

$$\omega_{\Delta AMB} = \sin^{-1}\left\{\frac{BM}{\sqrt{(BM)^2+(AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BM}{\sqrt{(BM)^2+(AM)^2}}\right)\left(\frac{P_1A}{\sqrt{(P_1A)^2+(AM)^2}}\right)\right\}$$

Similarly, solid angle subtended by the right ΔAMC at the the same point $P_1(0,0,h)$

$$\omega_{\Delta AMC} = \sin^{-1}\left\{\frac{CM}{\sqrt{(CM)^2+(AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CM}{\sqrt{(CM)^2+(AM)^2}}\right)\left(\frac{P_1A}{\sqrt{(P_1A)^2+(AM)^2}}\right)\right\}$$

Hence, the solid angle subtended by right ΔBAC at the the given point $\boldsymbol{P_1(0,0,h)}$ is calculated as

$$\Rightarrow \quad \boldsymbol{\omega_{\Delta BAC} = \omega_{\Delta AMB} + \omega_{\Delta AMC}}$$

Now, for the location of given point $\boldsymbol{P_2(0,0,h)}$ lying at a normal height h from the vertex 'B' (i.e. foot of perpendicular drawn from the point $P_2(0,0,h)$ to the plane of rightΔBAC) (See figure 6 above)

Using standard formula-1 & setting $b = AC, p = AB$ &

$h = normal\ height\ of\ point\ P_2(0,0,h) from\ the\ vertex\ 'B'(i.e. F.O.P.) = P_2B$

We get the solid angle subtended by the right ΔBAC at the the given point $P_2(0,0,h)$ as follows

$$\omega_{\Delta BAC} = \sin^{-1}\left\{\frac{AC}{\sqrt{(AC)^2+(AB)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AC}{\sqrt{(AC)^2+(AB)^2}}\right)\left(\frac{P_2B}{\sqrt{(P_2B)^2+(AB)^2}}\right)\right\}$$

Similarly, for the location of given point $\boldsymbol{P_3(0,0,h)}$ lying at a normal height h from the vertex 'C' (i.e. foot of perpendicular drawn from the point $P_3(0,0,h)$ to the plane of rightΔBAC) (See figure 6 above)

Using standard formula-1 & setting $b = AB, p = AC$ &

$h = normal\ height\ of\ point\ P_3(0,0,h) from\ the\ vertex\ 'C'(i.e. F.O.P.) = P_3C$

We get the solid angle subtended by the right ΔBAC at the the given point $P_3(0,0,h)$ as follows

$$\omega_{\Delta BAC} = \sin^{-1}\left\{\frac{AB}{\sqrt{(AB)^2+(AC)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AB}{\sqrt{(AB)^2+(AC)^2}}\right)\left(\frac{P_3C}{\sqrt{(P_3C)^2+(AC)^2}}\right)\right\}$$

- **Obtuse Angled Triangle:**

Consider the given point $\boldsymbol{P_1(0,0,h)}$ at a normal height h from the vertex 'A' (i.e. foot of perpendicular drawn from the point $P_1(0,0,h)$ to the plane of ΔABC) (See the figure 7)

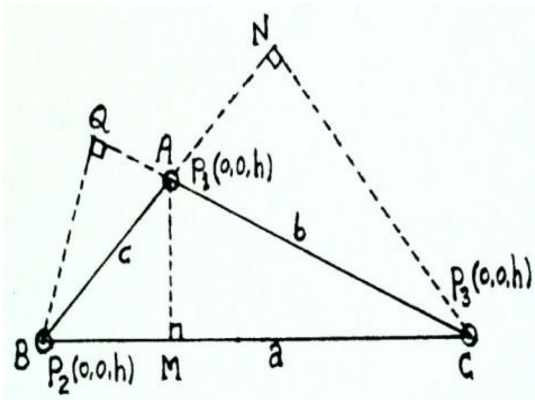


Fig 7: Obtuse Angled Triangle

Now, draw the perpendicular AM from F.O.P. 'A' to the opposite side BC to divide the obtuse ΔABC into two sub-elementary right triangles ΔAMB & ΔAMC

In this case, the area $(A_{\Delta ABC})$ of obtuse ΔABC is given by

$A_{\Delta ABC} =$
$algebraic\ sum\ of\ areas\ of\ elemetary\ triangles$

$$= A_{\Delta AMB} + A_{\Delta AMC} \qquad (\boldsymbol{obtuse\ \Delta ABC\ is\ internally\ divided})$$

Hence replacing the areas by the corresponding values of solid angles, we get

$\omega_{\Delta ABC} = algebraic\ sum\ of\ solid\ angles\ subtended\ by\ elemetary\ triangles\ at\ point\ P_1(0,0,h)$

$$= \omega_{\Delta AMB} + \omega_{\Delta AMC} \qquad (\boldsymbol{obtuse\ \Delta ABC\ is\ internally\ divided})$$

The values of $\omega_{\Delta AMB}$ & $\omega_{\Delta AMC}$ subtended by the right triangles ΔAMB & ΔAMC respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

Measure the dimensions AM, BM & P_1A from the drawing & set $b = BM, p = AM$ &

$h = normal\ height\ of\ point\ P_1(0,0,h) from\ the\ vertex\ 'A'(i.e. F.O.P.) = P_1A$ in above expression

We get, the solid angle subtended by the right ΔAMB at the given point $P_1(0,0,h)$

$$\omega_{\Delta AMB} = \sin^{-1}\left\{\frac{BM}{\sqrt{(BM)^2+(AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BM}{\sqrt{(BM)^2+(AM)^2}}\right)\left(\frac{P_1A}{\sqrt{(P_1A)^2+(AM)^2}}\right)\right\}$$

Similarly, solid angle subtended by the right ΔAMC at the the same point $P_1(0,0,h)$

$$\omega_{\Delta AMC} = \sin^{-1}\left\{\frac{CM}{\sqrt{(CM)^2+(AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CM}{\sqrt{(CM)^2+(AM)^2}}\right)\left(\frac{P_1A}{\sqrt{(P_1A)^2+(AM)^2}}\right)\right\}$$

Hence, the solid angle subtended by obtuse ΔABC at the the given point $\boldsymbol{P_1(0,0,h)}$ is calculated as

$$\Rightarrow \ \boldsymbol{\omega_{\Delta ABC} = \omega_{\Delta AMB} + \omega_{\Delta AMC}}$$

Now, for the location of given point $\boldsymbol{P_2(0,0,h)}$ lying at a normal height h from the vertex 'B' (i.e. foot of perpendicular drawn from the point $P_2(0,0,h)$ to the plane of ΔABC) (See figure 7 above)

Draw the perpendicular BQ from F.O.P. 'B' to the opposite side AC to divide the obtuse ΔABC into two sub-elementary right triangles ΔBQC & ΔBQA

In this case, the area $(A_{\Delta ABC})$ of obtuse ΔABC is given by

$$A_{\Delta ABC} = algebraic\ sum\ of\ areas\ of\ elemetary\ triangles$$

$$= A_{\Delta BQC} - A_{\Delta BQA} \qquad (\boldsymbol{obtuse\ \Delta ABC\ is\ externally\ divided})$$

Hence replacing the areas by the corresponding values of solid angles, we get

$$\omega_{\Delta ABC} = algebraic\ sum\ of\ solid\ angles\ subtended\ by\ elemetary\ triangles\ at\ point\ P_2(0,0,h)$$

$$= \omega_{\Delta BQC} - \omega_{\Delta BQA} \qquad (\boldsymbol{obtuse\ \Delta ABC\ is\ externally\ divided})$$

The values of $\omega_{\Delta BQC}$ & $\omega_{\Delta BQA}$ subtended by the right triangles ΔBQC & ΔBQA respectively are calculated by using standard formula-1 of right triangle (from eq(1) or eq(2)) as follows

$$\omega = \sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

Measure the dimensions QC, BQ & P_2B from the drawing & set $b = QC, p = BQ$ &

$h = normal\ height\ of\ point\ P_2(0,0,h) from\ the\ vertex\ 'B'(i.e. F.O.P.) = P_2B$ in above expression

We get, the solid angle subtended by the right ΔBQC at the given point $P_2(0,0,h)$

$$\omega_{\Delta BQC} = \sin^{-1}\left\{\frac{QC}{\sqrt{(QC)^2+(BQ)^2}}\right\} - \sin^{-1}\left\{\left(\frac{QC}{\sqrt{(QC)^2+(BQ)^2}}\right)\left(\frac{P_2B}{\sqrt{(P_2B)^2+(BQ)^2}}\right)\right\}$$

Similarly, solid angle subtended by the right ΔBQA at the the same point $P_2(0,0,h)$

$$\omega_{\Delta BQA} = \sin^{-1}\left\{\frac{QA}{\sqrt{(QA)^2+(BQ)^2}}\right\} - \sin^{-1}\left\{\left(\frac{QA}{\sqrt{(QA)^2+(BQ)^2}}\right)\left(\frac{P_2B}{\sqrt{(P_2B)^2+(BQ)^2}}\right)\right\}$$

Hence, the solid angle subtended by obtuse ΔABC at the the given point $\boldsymbol{P_2(0,0,h)}$ is calculated as

$$\Rightarrow \ \boldsymbol{\omega_{\Delta ABC} = \omega_{\Delta BQC} - \omega_{\Delta BQA}}$$

Similarly, for the location of given point $\boldsymbol{P_3(0,0,h)}$ lying at a normal height h from the vertex 'C' (i.e. foot of perpendicular drawn from the point $P_3(0,0,h)$ to the plane of ΔABC) (See figure 7 above)

Following the above procedure, solid angle subtended by obtuse ΔABC at the the given point $\boldsymbol{P_3(0,0,h)}$ is calculated as

$$\Rightarrow\ \boldsymbol{\omega_{\Delta ABC} = \omega_{\Delta CNB} - \omega_{\Delta CNA}} \quad (\boldsymbol{obtuse\ \Delta ABC\ is\ externally\ divided})$$

$$where,\ \ \omega_{\Delta CNB} = \sin^{-1}\left\{\frac{BN}{\sqrt{(BN)^2 + (CN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BN}{\sqrt{(BN)^2 + (CN)^2}}\right)\left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CN)^2}}\right)\right\}\ \ \&$$

$$\Rightarrow \omega_{\Delta CNA} = \sin^{-1}\left\{\frac{AN}{\sqrt{(AN)^2 + (CN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AN}{\sqrt{(AN)^2 + (CN)^2}}\right)\left(\frac{P_3C}{\sqrt{(P_3C)^2 + (CN)^2}}\right)\right\}$$

Now, we will apply the same procedure in case of any polygonal plane by diving it into the elementary triangles with common vertex at the F.O.P. & then each elementary triangle into two sub-elementary right triangles with common vertex at the F.O.P. Now, it's easy to understand the theory of polygon

VI. Axiom of polygon

"For a given point in the space, each of the polygons can be divided internally or externally or both w.r.t. foot of perpendicular (F.O.P.) drawn from the given point to the plane of polygon into a certain number of the elementary triangles, having a common vertex at the foot of perpendicular, by joining all the vertices of polygon to the F.O.P. by straight lines (generally extended)."

- Polygon is **externally divided** if the **F.O.P. lies outside the boundary**
- Polygon is divided **internally or externally or both** if the **F.O.P. lies inside or on the boundary** depending on the **geometrical shape of polygon (i.e. angles & sides)**

(See the different cases in the figures (8), (9), (10) & (11) below as explained in Theory of Polygon)

Elementary Triangle: Any of the elementary triangles obtained by joining all the vertices of polygon to the foot of perpendicular drawn from the given point to the plane of polygon such that all the elementary triangles have common vertex at the foot of perpendicular (**F.O.P.**) is called elementary triangle.

Sub-elementary Right Triangles: These are the right triangles which are obtained by drawing a perpendicular from F.O.P. to the opposite side in any of the elementary triangles. Thus one elementary triangle is internally or externally divided into two sub-elementary right triangles. These sub-elementary right triangles always have common vertex at the foot of perpendicular (F.O.P.)

VII. HCR's Theory of Polygon

(Proposed by the Author-2014)

This theory is applicable for any polygonal plane (i.e. **plane bounded by the straight lines only**) & any point in the space if the following parameters are already known

1. Geometrical shape & dimensions of the polygonal plane

2. Normal distance (h) of the given point from the plane of polygon

3. Location of foot of perpendicular (F.O.P.) drawn from given point to the plane of polygon

According to this theory **"Solid angle subtended by any polygonal plane at any point in the space is the algebraic sum of solid angles subtended at the same point by all the elementary triangles (obtained by joining all the vertices of polygon to the foot of perpendicular) having common vertex at the foot of perpendicular drawn from the given point to the plane of polygon such that algebraic sum of areas of all these triangles is equal to the area of given polygon."** It has no mathematical proof.

Mathematically, solid angle $(\omega_{polygon})$ subtended by any polygonal plane with 'n' number of sides & area 'A' at any point in the space is given by

$$\omega_{polygon} = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 \dots\dots\dots\dots\dots\dots$$

$$\Rightarrow \boldsymbol{\omega_{polygon}} = \left[\sum \boldsymbol{\omega_i}\right]_{algebraic}$$

$$where, \quad A_{polygon} = A_1 + A_2 + A_3 + A_4 + A_5 \dots\dots\dots\dots\dots\dots\dots. = \left[\sum \boldsymbol{A_i}\right]_{algebraic}$$

$A_1, A_2, A_3, A_4, A_5, \dots \dots.$ are the areas of elementary triangles subtending the solid angles $\omega_1, \omega_2, \omega_3, \omega_4. \omega_5 \dots \dots$ at the given point which are determined by using standard formula-1 (as given from eq(1)) along with the necessary dimensions which are found out analytically or by tracing the diagram which is easier.

- **Element Method:** It the method of **diving a polygon internally** or **externally** into **sub-elementary right triangles** having **common vertex at the F.O.P.** drawn from the given point to the plane of polygon such that **algebraic sum of areas** of all these triangles is equal to the **area of given polygon**."

- **Working Steps:**

STEP 1: Trace/draw the diagram of the given polygon (plane) with known geometrical dimensions.

STEP 2: Specify the location of foot of perpendicular drawn from a given point to the plane of polygon.

STEP 3: Join all the vertices of polygon to the foot of perpendicular by the extended straight lines. Thus the polygon is divided into a number of elementary triangles having a common vertex at the foot of perpendicular.

STEP 4: Further, consider each elementary triangle & divide it internally or externally into two sub-elementary right triangles simply by drawing a perpendicular from F.O.P. (i.e. common vertex) to the opposite side of that elementary triangle.

STEP 5: Now, find out the area of given polygon as the algebraic sum of areas of all these sub-elementary right triangles i.e. area of each of right triangles must be taken with proper sign whose sum gives area of polygon.

Remember: All the elementary triangles & sub-elementary right triangles must have their one vertex common at the foot of perpendicular (F.O.P.).

STEP 6: Replace areas of all these sub-elementary right triangles by their respective values of solid angle subtended at the given point in the space.

STEP 7: Calculate solid angle subtended by each of individual sub-elementary right triangles by using the standard formula-1 of right triangle (from eq(1) as derived above) as follows

$$\omega = \sin^{-1}\left\{\frac{b}{\sqrt{b^2+p^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

STEP 8: Now, solid angle subtended by polygonal plane at the given point will be the algebraic sum of all its individual sub-elementary right triangles as given (By **Element Method**)

$$\omega_{polygon} = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \ \dots\dots\dots\dots$$

*** Finally**, we get the value of $\boldsymbol{\omega_{polygon}}$ as the **algebraic sum** of solid angles subtended by **right triangles only**.

VIII. Special Cases for a Polygonal Plane

Let's us consider a polygonal plane (with vertices) 123456 & a given point say $P(0, 0, h)$ at a normal height h from the given plane in the space.

Now, specify the location of foot of perpendicular say 'O' on the plane of polygon which may lie

1. **Outside the boundary**
2. **Inside the boundary**
3. **On the boundary**

a. On one of the sides or b. At one of the vertices

Let's consider the above cases one by one as follows

1. F.O.P. outside the boundary:

Let the foot of perpendicular 'O' lie outside the boundary of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the extended straight lines

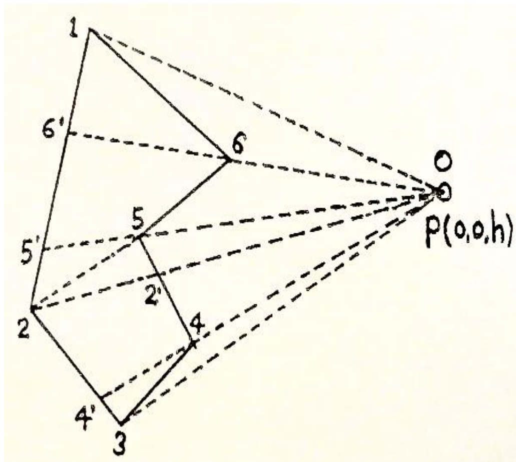


Fig 8: F.O.P. lying outside the boundary

(As shown in the figure 8) Thus the polygon is divided into elementary triangles (obtained by the extension lines), all having common vertex at the foot of perpendicular 'O'. Now the solid angle subtended by the polygonal plane at the given point 'P' in the space is given

By Element-Method

$$\omega_{123456} = \omega_{616'} + \omega_{566'5'} + \omega_{22'55'} + \omega_{4'42'2} + \omega_{344'} \qquad \dots\dots\dots\dots\dots\dots\dots.(I)$$

$$where, \quad \omega_{616'} = \omega_{6'O1} - \omega_{6O1}$$

$$\omega_{566'5'} = \omega_{5'O6'} - \omega_{5O6}\omega_{22'55'} = \omega_{2O5'} - \omega_{2'O5}$$

$$\omega_{4'42'2} = \omega_{4'O2} - \omega_{4O2'}$$

$$\omega_{344'} = \omega_{3O4'} - \omega_{3O4}$$

Thus, the value of solid angle subtended by the polygon at the given point is obtained by setting these values in the eq. (I) as follows

$$\Rightarrow \qquad \omega_{123456} = (\omega_{6'O1} - \omega_{6O1}) + (\omega_{5'O6'} - \omega_{5O6}) + (\omega_{2O5'} - \omega_{2'O5}) + (\omega_{4'O2} - \omega_{4O2'})$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

2. F.O.P. inside the boundary:

Let the foot of perpendicular 'O' lie inside the boundary of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the extended straight lines (As shown in the figure 9)

Thus the polygon is divided into elementary triangles (obtained by the extension lines), all having common vertex at the foot of perpendicular 'O'. Now the solid angle subtended by the polygonal plane at the given point P in the space is given

By Element-Method

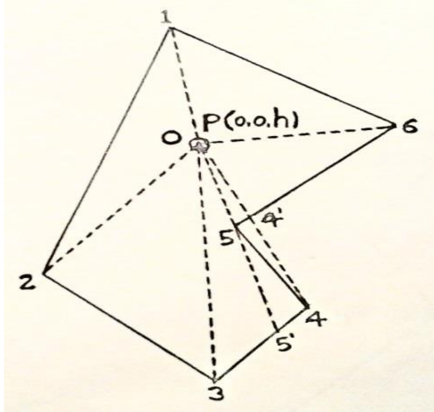


Figure 9: F.O.P. lying inside the boundary

$$\omega_{123456} = \omega_{102} + \omega_{203} + \omega_{305'} + \omega_{5'54} + \omega_{504'} \quad \dots\dots\dots\dots\dots\dots\dots.(II)$$

Where,

$$\omega_{5'54} = \omega_{5'04} - \omega_{504}$$

Thus, the value of solid angle subtended by the polygon at the given point is obtained by setting these values in the eq. (II) as follows

$$\Rightarrow \quad \omega_{123456} = \omega_{102} + \omega_{203} + \omega_{305'} + (\omega_{5'04} - \omega_{504}) + \omega_{504'}$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

3. F.O.P. on the boundary: Further two cases are possible

a. F.O.P. lying on one of the sides:

Let the foot of perpendicular 'O' lie on one of the sides say '12' of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular 'O' by the straight lines

(As shown in the figure 10)

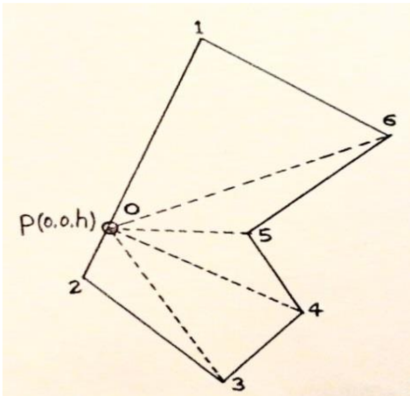


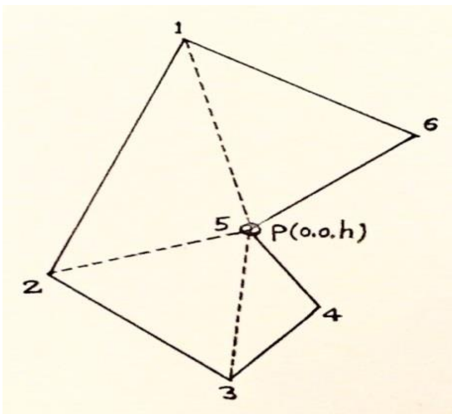
Figure 10: F.O.P. lying at one of the sides

Thus the polygon is divided into elementary triangles (obtained by the straight lines), all having common vertex at the foot of perpendicular 'O'. Now the solid angle subtended by the polygonal plane at the given point 'P' in the space is given By Element-Method

$$\omega_{123456} = \omega_{106} + \omega_{605} + \omega_{504} + \omega_{403} + \omega_{302} \quad \dots\dots\dots\dots\dots\dots\dots.(III)$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

b. F.O.P. lying at one of the vertices:



Let the foot of perpendicular lie on one of the vertices say ‘5’ of polygon. Join all the vertices of polygon (plane) 123456 to the foot of perpendicular (i.e. common vertex ‘5’) by the straight lines

(As shown in the figure 11)

Thus the polygon is divided into elementary triangles (obtained by the straight lines), all having common vertex at the foot of perpendicular ‘5’. Now the solid angle subtended by the polygonal plane at the given point ‘P’ in the space is given

By Element-Method

Fig 11: F.O.P. lying on one of the vertices

$$\omega_{123456} = \omega_{152} + \omega_{253} + \omega_{354} + \omega_{156} \qquad \dots\dots\dots\dots\dots\dots\dots.(IV)$$

Further, each of the individual elementary triangles is divided into two sub-elementary right triangles for which the values of solid angle are determined by using standard formula-1 of right triangle.

IX. Analytical Applications of Theory of Polygon

Let’s take some examples of polygonal plane to find out the solid angle at different locations of a given point in the space. For ease of understanding, we will take some particular examples where division of polygons into right triangles is easier & standard formula can directly be applied by analytical measurements of necessary dimensions i.e. drawing is not required. Although, random location of point may cause complex calculations

- **Right Triangular Plane**

F.O.P. lying on the right angled vertex: Let there be a right triangular plane ABC having orthogonal sides AB = a & BC = b & a given point say P(0, 0, h) at a normal height h from the right angled vertex ‘B’

(As shown in the figure 12 below)

Now, draw a perpendicular BN from the vertex ‘B’ to the hypotenuse AC to divide the right ΔABC into elementary right triangles ΔANB & ΔBNC. By element method, solid angle subtended by the right ΔABC at the given point P

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta ANB} + \omega_{\Delta BNC}$$

Where, the values of $\omega_{\Delta ANB}$ & $\omega_{\Delta BNC}$ are calculated by standard formula-1 as follows

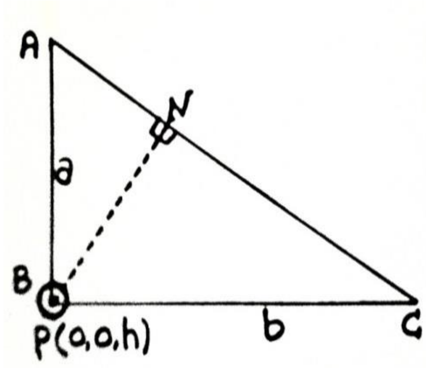


Fig 12: Point P lying on the normal axis passing through the right angled vertex B

Values of BN, AN & CN can be easily calclated as follows

$$BN = \frac{ab}{\sqrt{a^2+b^2}}, \quad AN = \frac{a^2}{\sqrt{a^2+b^2}} \quad \& \quad CN = \frac{b^2}{\sqrt{a^2+b^2}}$$

$$\therefore \ \omega_{\Delta ANB} = \sin^{-1}\left\{\frac{AN}{\sqrt{(AN)^2+(BN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AN}{\sqrt{(AN)^2+(BN)^2}}\right)\left(\frac{PB}{\sqrt{(PB)^2+(BN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\frac{a^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2+\left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\frac{a^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{a^2}{\sqrt{a^2+b^2}}\right)^2+\left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}}\right)\left(\frac{h}{\sqrt{(h)^2+\left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}}\right)\right\}$$

$$\Rightarrow \omega_{\Delta ANB} = \sin^{-1}\left\{\frac{a}{\sqrt{a^2+b^2}}\right\} - \sin^{-1}\left\{\frac{ah}{\sqrt{h^2(a^2+b^2)+a^2b^2}}\right\}$$

Similarly, we can calculate solid angle subtended by the ΔCNB at the given point $P(0,0,h)$

$$\omega_{\Delta CNB} = \sin^{-1}\left\{\frac{CN}{\sqrt{(CN)^2+(BN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CN}{\sqrt{(CN)^2+(BN)^2}}\right)\left(\frac{PB}{\sqrt{(PB)^2+(BN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\frac{b^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{b^2}{\sqrt{a^2+b^2}}\right)^2+\left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\frac{b^2}{\sqrt{a^2+b^2}}}{\sqrt{\left(\frac{b^2}{\sqrt{a^2+b^2}}\right)^2+\left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}}\right)\left(\frac{h}{\sqrt{(h)^2+\left(\frac{ab}{\sqrt{a^2+b^2}}\right)^2}}\right)\right\}$$

$$\Rightarrow \omega_{\Delta CNB} = \sin^{-1}\left\{\frac{b}{\sqrt{a^2+b^2}}\right\} - \sin^{-1}\left\{\frac{bh}{\sqrt{h^2(a^2+b^2)+a^2b^2}}\right\}$$

Hence, the solid angle subtended by given right ΔABC at the given point $P(0,0,h)$ lying at a normal height h from the right angled vertex 'B' is calculated as follows

$$\Rightarrow \omega_{\Delta ABC} = \omega_{\Delta ANB} + \omega_{\Delta CNB}$$

$$= \sin^{-1}\left\{\frac{a}{\sqrt{a^2+b^2}}\right\} - \sin^{-1}\left\{\frac{ah}{\sqrt{h^2(a^2+b^2)+a^2b^2}}\right\} + \sin^{-1}\left\{\frac{b}{\sqrt{a^2+b^2}}\right\} - \sin^{-1}\left\{\frac{bh}{\sqrt{h^2(a^2+b^2)+a^2b^2}}\right\}$$

$$= \left[\sin^{-1}\left\{\frac{a}{\sqrt{a^2+b^2}}\right\} + \sin^{-1}\left\{\frac{b}{\sqrt{a^2+b^2}}\right\}\right] - \left[\sin^{-1}\left\{\frac{ah}{\sqrt{h^2(a^2+b^2)+a^2b^2}}\right\} + \sin^{-1}\left\{\frac{bh}{\sqrt{h^2(a^2+b^2)+a^2b^2}}\right\}\right]$$

Using, $\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\ \forall\ (-1 \le (x,y) \le 1)$ & simplifying, we get

$$\Rightarrow \omega_{\Delta ABC} = \sin^{-1}\left\{\frac{a^2+b^2}{a^2+b^2}\right\} - \sin^{-1}\left\{\frac{h\left(a^2\sqrt{h^2+b^2} + b^2\sqrt{h^2+a^2}\right)}{h^2(a^2+b^2)+a^2b^2}\right\}$$

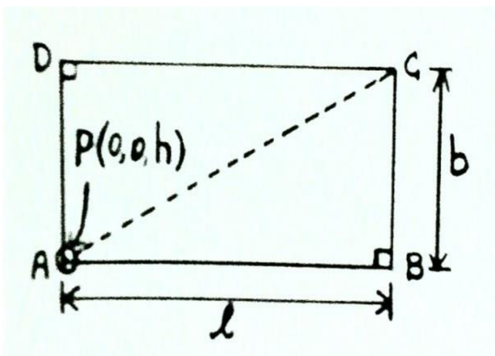
$$= \frac{\pi}{2} - \sin^{-1}\left\{\frac{h\left(a^2\sqrt{h^2+b^2} + b^2\sqrt{h^2+a^2}\right)}{h^2(a^2+b^2)+a^2b^2}\right\}$$

$$\therefore\ \boldsymbol{\omega_{\Delta ABC} = \cos^{-1}\left\{\frac{h\left(a^2\sqrt{h^2+b^2} + b^2\sqrt{h^2+a^2}\right)}{h^2(a^2+b^2)+a^2b^2}\right\}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Note: This is the standard formula to find out the value of solid angle subtended by a right triangular plane, with orthogonal sides $\boldsymbol{a}$ & $\boldsymbol{b}$, at any point lying at a normal height h from the right angled vertex.

- **Rectangular Plane**

F.O.P. lying on one of the vertices of rectangular plane: Let there be a rectangular plane ABCD having length $AB = l$ & width $BC = b$



& a given point say P(0, 0, h) at a normal height h from any of the vertices say vertex 'A'

Now, draw a perpendicular PA from the given point 'P' to the plane of rectangle ABCD passing through the vertex A (i.e. foot of perpendicular). Join the vertex 'C' to the F.O.P. 'A' to divide the plane into elementary triangles I.e. right triangles ΔABC & ΔADC.

It is clear from the figure (13) that the area of rectangle ABCD

$A_{ABCD} = A_{\Delta ABC} + A_{\Delta ADC}$

Hence, using **Element Method** by replacing areas by corresponding values of solid angles, the solid angle subtended by the rectangular plane ABCD at the given point P

Fig 13: Point P lying on the normal axis passing through one of the vertices of rectangular plane

$\omega_{ABCD} = \omega_{\Delta ABC} + \omega_{\Delta ADC}$

Now, using standard formula-1, solid angle subtended by the right ΔABC at the point $P(0, 0, h)$

$$\omega_{\Delta ABC} = \sin^{-1}\left\{\frac{BC}{\sqrt{(BC)^2 + (AB)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BC}{\sqrt{(BC)^2 + (AB)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2 + (AB)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{b}{\sqrt{b^2 + l^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2 + l^2}}\right)\left(\frac{h}{\sqrt{h^2 + l^2}}\right)\right\} \qquad \&$$

$$\omega_{\Delta ADC} = \sin^{-1}\left\{\frac{CD}{\sqrt{(CD)^2 + (AD)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CD}{\sqrt{(CD)^2 + (AD)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2 + (AD)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{l}{\sqrt{l^2 + b^2}}\right\} - \sin^{-1}\left\{\left(\frac{l}{\sqrt{l^2 + b^2}}\right)\left(\frac{h}{\sqrt{h^2 + b^2}}\right)\right\}$$

Hence, the solid angle subtended by rectangular plane ABCD at the given point $P(0, 0, h)$lying at a normal height h from vertex A

$$\omega_{ABCD} = \omega_{\Delta ABC} + \omega_{\Delta ADC}$$

$$= \sin^{-1}\left\{\frac{b}{\sqrt{b^2 + l^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2 + l^2}}\right)\left(\frac{h}{\sqrt{h^2 + l^2}}\right)\right\} + \sin^{-1}\left\{\frac{l}{\sqrt{l^2 + b^2}}\right\} - \sin^{-1}\left\{\left(\frac{l}{\sqrt{l^2 + b^2}}\right)\left(\frac{h}{\sqrt{h^2 + b^2}}\right)\right\}$$

$$= \left[\sin^{-1}\left\{\frac{l}{\sqrt{l^2 + b^2}}\right\} + \sin^{-1}\left\{\frac{b}{\sqrt{b^2 + l^2}}\right\}\right] - \left[\sin^{-1}\left\{\left(\frac{l}{\sqrt{l^2 + b^2}}\right)\left(\frac{h}{\sqrt{h^2 + b^2}}\right)\right\} + \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2 + l^2}}\right)\left(\frac{h}{\sqrt{h^2 + l^2}}\right)\right\}\right]$$

Using, $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1 - y^2} + y\sqrt{1 - x^2}\right) \ \forall \ (-1 \leq (x, y) \leq 1)$ & simplifying, we get

$$\Rightarrow \omega_{ABCD} = \sin^{-1}\left\{\frac{l^2 + b^2}{l^2 + b^2}\right\} - \sin^{-1}\left\{\frac{h(l^2 + b^2)\sqrt{l^2 + b^2 + h^2}}{(l^2 + b^2)\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}}\right\}$$

$$= \frac{\pi}{2} - \sin^{-1}\left\{\frac{h\sqrt{l^2 + b^2 + h^2}}{\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}}\right\} = \cos^{-1}\left\{\frac{h\sqrt{l^2 + b^2 + h^2}}{\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}}\right\} = \sin^{-1}\left\{\frac{lb}{\sqrt{l^2 + h^2}\sqrt{b^2 + h^2}}\right\}$$

$$\therefore \ \boldsymbol{\omega_{ABCD} = \sin^{-1}\left\{\frac{lb}{\sqrt{(l^2 + h^2)(b^2 + h^2)}}\right\}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Note: This is the standard formula to find out the value of solid angle subtended by a rectangular plane of size $a \times b$ at any point lying at a normal height h from any of the vertices.

F.O.P. lying on the centre of rectangular plane: Let the given point P(0, 0, h) be lying at a normal height h from the centre 'O' (i.e. F.O.P.) of rectangular plane ABCD. Join all the vertices A, B, C, D to the F.O.P. 'O'.

Thus, rectangle ABCD is divided into four elementary triangles $\Delta AOB, \Delta BOC, \Delta COD$ & ΔAOD. Hence, Area of rectangle ABCD

$$\Rightarrow A_{ABCD} = A_{\Delta AOB} + A_{\Delta BOC} + A_{\Delta COD} + A_{\Delta AOD}$$

$$= 2(A_{\Delta AOB} + A_{\Delta AOD}) \qquad By\ symmetry$$

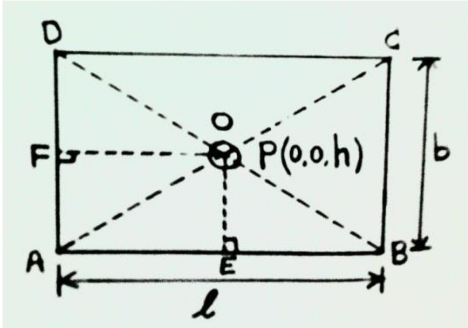


Fig 14: Point P lying on the normal axis passing through the centre of rectangular plane

Now, draw perpendiculars OE & OF from the F.O.P. 'O' to the opposite sides AB in ΔAOB & AD in ΔAOD to divide them into sub-elementary right triangles $\Delta OEA, \Delta OEB$ in ΔAOB &

$\Delta OFA, \Delta OFD$ in ΔAOD

$$\Rightarrow A_{ABCD} = 2 \times 2(A_{\Delta OEA} + A_{\Delta OFA}) = 4(A_{\Delta OEA} + A_{\Delta OFA}) \qquad By\ symmetry$$

Hence, using **Element Method** by replacing areas by corresponding values of solid angles, the solid angle subtended by the rectangular plane ABCD at the given point P

$$\omega_{ABCD} = 4(\omega_{\Delta OEA} + \omega_{\Delta OFA})$$

Now, using standard formula-1, solid angle subtended by the right ΔOEA at the point $P(0, 0, h)$

$$\omega_{\Delta OEA} = \sin^{-1}\left\{\frac{AE}{\sqrt{(AE)^2 + (OE)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AE}{\sqrt{(AE)^2 + (OE)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2 + (OE)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\left(\frac{l}{2}\right)}{\sqrt{\left(\frac{l}{2}\right)^2 + \left(\frac{b}{2}\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\left(\frac{l}{2}\right)}{\sqrt{\left(\frac{l}{2}\right)^2 + \left(\frac{b}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2 + \left(\frac{b}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{l}{\sqrt{l^2 + b^2}}\right\} - \sin^{-1}\left\{\left(\frac{l}{\sqrt{l^2 + b^2}}\right)\left(\frac{2h}{\sqrt{4h^2 + b^2}}\right)\right\} \quad \&$$

$$\omega_{\Delta OFA} = \sin^{-1}\left\{\frac{AF}{\sqrt{(AF)^2 + (OF)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AF}{\sqrt{(AF)^2 + (OF)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{l}{2}\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{l}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2 + \left(\frac{l}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{b}{\sqrt{l^2 + b^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{l^2 + b^2}}\right)\left(\frac{2h}{\sqrt{4h^2 + l^2}}\right)\right\}$$

Hence, the solid angle subtended by rectangular plane ABCD at the given point $P(0, 0, h)$lying at a normal height h from vertex A

$$\omega_{ABCD} = 4(\omega_{\Delta OEA} + \omega_{\Delta OFA})$$

$$= 4\left[\sin^{-1}\left\{\frac{l}{\sqrt{l^2 + b^2}}\right\} - \sin^{-1}\left\{\left(\frac{l}{\sqrt{l^2 + b^2}}\right)\left(\frac{2h}{\sqrt{4h^2 + b^2}}\right)\right\} + \sin^{-1}\left\{\frac{b}{\sqrt{l^2 + b^2}}\right\} - \sin^{-1}\left\{\left(\frac{b}{\sqrt{l^2 + b^2}}\right)\left(\frac{2h}{\sqrt{4h^2 + l^2}}\right)\right\}\right]$$

$$= 4\left[\sin^{-1}\left\{\frac{l}{\sqrt{l^2+b^2}}\right\} + \sin^{-1}\left\{\frac{b}{\sqrt{l^2+b^2}}\right\}\right] - 4\left[\sin^{-1}\left\{\left(\frac{l}{\sqrt{l^2+b^2}}\right)\left(\frac{2h}{\sqrt{4h^2+b^2}}\right)\right\} + \sin^{-1}\left\{\left(\frac{b}{\sqrt{l^2+b^2}}\right)\left(\frac{2h}{\sqrt{4h^2+l^2}}\right)\right\}\right]$$

Using, $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\ \forall\ (-1 \le (x, y) \le 1)$ & simplifying, we get

$$\Rightarrow \omega_{ABCD} = 4\left[\sin^{-1}\left\{\frac{l^2+b^2}{l^2+b^2}\right\} - \sin^{-1}\left\{\frac{2h(l^2\ +\ b^2)\sqrt{l^2+b^2+4h^2}}{(l^2+b^2)\sqrt{l^2+4h^2}\sqrt{b^2+4h^2}}\right\}\right]$$

$$= 4\left[\frac{\pi}{2} - \sin^{-1}\left\{\frac{2h\sqrt{l^2+b^2+4h^2}}{\sqrt{l^2+4h^2}\sqrt{b^2+4h^2}}\right\}\right]$$

$$= 4\cos^{-1}\left\{\frac{2h\sqrt{l^2+b^2+4h^2}}{\sqrt{l^2+4h^2}\sqrt{b^2+4h^2}}\right\} = 4\sin^{-1}\left\{\frac{lb}{\sqrt{l^2+4h^2}\sqrt{b^2+4h^2}}\right\}$$

$$\therefore\ \boldsymbol{\omega_{ABCD} = 4\sin^{-1}\left\{\frac{lb}{\sqrt{(l^2+4h^2)(b^2+4h^2)}}\right\}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

Note: **This is the standard formula to find out the value of solid angle subtended by a rectangular plane of size $\boldsymbol{a \times b}$ at any point lying at a normal height h from the centre.**

- **Rhombus-like Plane**

Let there be a rhombus-like plane ABCD having diagonals $AC = 2d_1$ & $BD = 2d_2$ bisecting each other at right angle at the centre 'O' and a given point say P(0, 0, h) at a height 'h' lying on the normal axis passing through the centre 'O' (i.e. foot of perpendicular) (See the figure 15)

Join all the vertices A, B, C & D to the F.O.P. to divide the rhombus ABCD into elementary right triangles $\Delta AOB, \Delta BOC, \Delta COD$ & ΔAOD. Now, area of rhombus ABCD

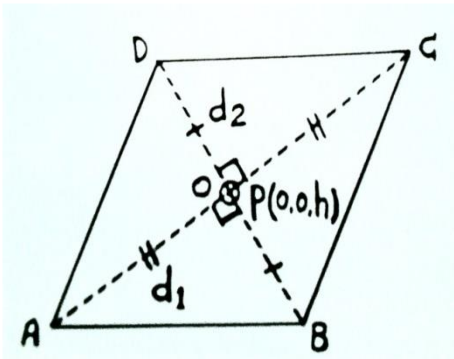


Fig 15: Point P lying on the normal axis passing through the centre of rhombus

$$\Rightarrow A_{ABCD} = A_{\Delta AOB} + A_{\Delta BOC} + A_{\Delta COD} + A_{\Delta AOD}$$

$$= 4(A_{\Delta AOB}) \qquad \textit{By symmetry}$$

Hence, using **Element Method** by replacing area by corresponding value of solid angle, the solid angle subtended by the rhombus-like plane ABCD at the given point P.

$$\omega_{ABCD} = 4\omega_{\Delta AOB}$$

From eq(3), we know that solid angle subtended by a right triangular plane at any point lying on the vertical passing though right angled vertex is given as

$$\omega = \cos^{-1}\left\{\frac{h\left(a^2\sqrt{h^2+b^2}\ +\ b^2\sqrt{h^2+a^2}\right)}{h^2(a^2+b^2)+a^2b^2}\right\}$$

On setting, $a = d_1$ & $b = d_2$ in the above equation, we get

$$\omega_{\Delta AOB} = \cos^{-1}\left\{\frac{h\left({d_1}^2\sqrt{h^2+{d_2}^2}\ +\ {d_2}^2\sqrt{h^2+{d_1}^2}\right)}{h^2\left({d_1}^2+{d_2}^2\right)+{d_1}^2{d_2}^2}\right\}$$

Hence, the solid angle subtended by rhombus-like plane ABCD at the given point $P(0,0,h)$lying at a normal height h from centre 'O'

$$\Rightarrow \boldsymbol{\omega_{ABCD} = 4\cos^{-1}\left\{\frac{h\left(d_1{}^2\sqrt{h^2+d_2{}^2}\ +\ d_2{}^2\sqrt{h^2+d_1{}^2}\right)}{h^2\left(d_1{}^2+d_2{}^2\right)+d_1{}^2d_2{}^2}\right\}} \quad \dots\dots\dots\dots\dots(6)$$

Note: This is the standard formula to find out the value of solid angle subtended by a rhombus-like plane having diagonals $2d_1$ & $2d_2$ at any point lying at a normal height h from the centre.

- **Regular Polygonal Plane**

Let there be a regular polygonal plane $A_1A_2A_3\dots\dots A_n$ having 'n' no. of the sides each equal to the length 'a' & a given point say P(0, 0, h) at a height 'h' lying on the normal height h from the centre 'O'. (i.e. foot of perpendicular) (See the figure 16)

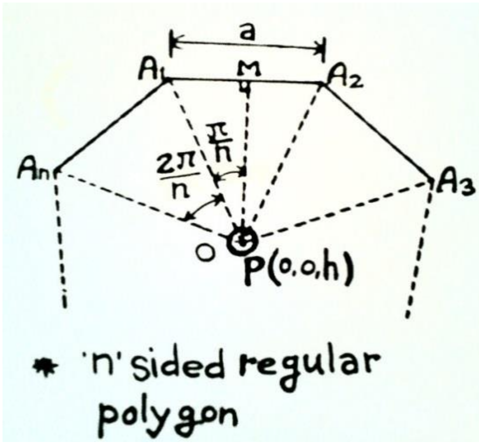


Fig 16: Point P lying on the normal axis passing through the centre of regular polygonal

Join all the vertices $A_1, A_2, A_3, \dots\dots \& A_n$ to the F.O.P. 'O' to divide the polygon into elementary triangles $\Delta OA_1A_2, \Delta OA_2A_3, \dots\dots\dots\dots\dots, \Delta OA_{n-1}A_n$ which are congruent hence the area of polygon

$$A_{reg.poly.} = A_{\Delta OA_1A_2} + A_{\Delta OA_2A_3} + A_{\Delta OA_3A_4} + \ \dots\dots\dots.+A_{\Delta OA_nA_1}$$

$$= n\left(A_{\Delta OA_1A_2}\right) = n\left(A_{\Delta OMA_1} + A_{\Delta OMA_2}\right)$$

Now, draw a perpendicular OM from F.O.P. 'O' to opposite side A_1A_2 to divide ΔOA_1A_2 into two sub-elementary right triangles ΔOMA_1 & ΔOMA_2 which are congruent. Hence area of polygon

$$A_{reg.poly.} = n \times 2\left(A_{\Delta OMA_1}\right) = 2n\left(A_{\Delta OMA_1}\right)$$

Hence, using **Element Method** by replacing area by corresponding value of solid angle, the solid angle subtended by the regular polygonal plane $A_1A_2A_3\dots\dots A_n$ at the given point P.

$$\omega_{ABCD} = 2n\left(\omega_{\Delta OMA_1}\right)$$

Now, using standard formula-1, solid angle subtended by the right ΔOMA_1 at the point $P(0,0,h)$

$$\omega_{\Delta OMA_1} = \sin^{-1}\left\{\frac{A_1M}{\sqrt{(A_1M)^2+(OM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{A_1M}{\sqrt{(A_1M)^2+(OM)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OM)^2}}\right)\right\}$$

Now, setting the corresponding values in above expression as follows

$$A_1M = \frac{a}{2}, \qquad OM = \frac{a}{2}\cot\frac{\pi}{n} \ \ \& PO = h, \ \ we\ have$$

$$\omega_{\Delta OMA_1} = \sin^{-1}\left\{\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\cot\frac{\pi}{n}\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}\cot\frac{\pi}{n}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2+\left(\frac{a}{2}\cot\frac{\pi}{n}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1}{cosec\frac{\pi}{n}}\right\} - \sin^{-1}\left\{\left(\frac{1}{cosec\frac{\pi}{n}}\right)\left(\frac{2h}{\sqrt{4h^2 + a^2cot^2\frac{\pi}{n}}}\right)\right\}$$

$$= \sin^{-1}\left\{sin\frac{\pi}{n}\right\} - \sin^{-1}\left\{\frac{2hsin\frac{\pi}{n}}{\sqrt{4h^2 + a^2cot^2\frac{\pi}{n}}}\right\} = \frac{\pi}{n} - \sin^{-1}\left\{\frac{2hsin\frac{\pi}{n}}{\sqrt{4h^2 + a^2cot^2\frac{\pi}{n}}}\right\}$$

Hence, the solid angle subtended by regular polygonal plane $A_1A_2A_3 \dots\dots A_n$ at the given point $P(0, 0, h)$ lying at a normal height h from centre 'O'

$$= 2n\left[\frac{\pi}{n} - \sin^{-1}\left\{\frac{2hsin\frac{\pi}{n}}{\sqrt{4h^2 + a^2cot^2\frac{\pi}{n}}}\right\}\right]$$

$$\Rightarrow \boldsymbol{\omega_{reg.poly.} = 2\pi - 2n\sin^{-1}\left\{\frac{2hsin\frac{\pi}{n}}{\sqrt{4h^2 + a^2cot^2\frac{\pi}{n}}}\right\} \quad \forall\, n \geq 3} \qquad \dots\dots\dots\dots\dots (7)$$

Note: This is the standard formula to find out the value of solid angle subtended by a regular polygonal plane, having n number of sides each of length a, at any point lying at a normal height h from the centre.

- **Regular Pentagonal Plane**

Let there be a regular pentagonal plane ABCDE having each side of length a and a given point say P(0, 0, h) lying at a normal height 'h' from the centre 'O' (i.e. foot of perpendicular) (See figure 17)

Join all the vertices A, B, C, D & E to the F.O.P. 'O' to divide the pentagon ABCDE into elementary triangles $\Delta ABC, \Delta ACD$ & ΔADE.

Now, draw perpendiculars AN, AQ & AM to the opposite sides BC, CD & DE in $\Delta ABC, \Delta ACD$ & ΔADE respectively to divide each elementary triangle into two right triangles. Hence the area of regular pentagon

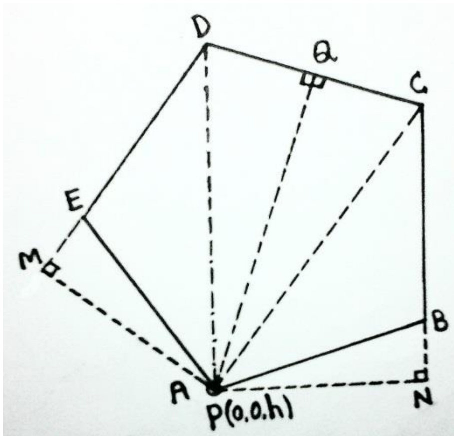


Fig 17: Point P lying on the normal axis passing through the vertex A of regular pentagon

$$A_{reg.penta} = 2(A_{\Delta ABC} + A_{\Delta AQC}) \quad by\ symmetry$$

$$= 2(A_{\Delta ANC} - A_{\Delta ANB} + A_{\Delta AQC}) \qquad (A_{\Delta ABC} = A_{\Delta ANC} - A_{\Delta ANB})$$

Hence, using **Element Method** by replacing areas by corresponding values of solid angle, the solid angle subtended by the regular pentagonal plane ABCDE at the given point P.

$$\omega_{reg.penta.} = 2(\omega_{\Delta ANC} - \omega_{\Delta ANB} + \omega_{\Delta AQC}) = 2(\omega_{\Delta ANC} + \omega_{\Delta AQC} - \omega_{\Delta ANB}) =$$

Necessary dimensions can be calculated by the figure as follows

$$AN = acos18^o, \qquad BN = asin18^o, \qquad CN = a + asin18^o \quad \& \quad AQ = \frac{a}{2}cot18^o$$

Now, using standard formula-1, solid angle subtended by the right ΔANC at the point $P(0,0,h)$

$$\omega_{\Delta ANC} = \sin^{-1}\left\{\frac{CN}{\sqrt{(CN)^2+(AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CN}{\sqrt{(CN)^2+(AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{a+asin18^o}{\sqrt{(a+asin18^o)^2+(acos18^o)^2}}\right\} - \sin^{-1}\left\{\left(\frac{a+asin18^o}{\sqrt{(a+asin18^o)^2+(acos18^o)^2}}\right)\left(\frac{h}{\sqrt{h^2+(acos18^o)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1+sin18^o}{\sqrt{2(1+sin18^o)}}\right\} - \sin^{-1}\left\{\left(\frac{1+sin18^o}{\sqrt{2(1+sin18^o)}}\right)\left(\frac{h}{\sqrt{h^2+a^2cos^2 18^o}}\right)\right\}$$

$$= \sin^{-1}\left\{\sqrt{\frac{1+sin18^o}{2}}\right\} - \sin^{-1}\left\{\left(\sqrt{\frac{1+sin18^o}{2}}\right)\left(\frac{h}{\sqrt{h^2+a^2cos^2 18^o}}\right)\right\}$$

$$\omega_{\Delta ANC} = \frac{3\pi}{10} - \sin^{-1}\left(\frac{hcos36^o}{\sqrt{h^2+a^2cos^2 18^o}}\right) \qquad \left(since,\ \sqrt{\frac{1+sin18^o}{2}} = cos36^o\right)$$

Similarly, we get solid angle subtended by right ΔANB at the given point 'P'

$$\omega_{\Delta ANB} = \sin^{-1}\left\{\frac{BN}{\sqrt{(BN)^2+(AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BN}{\sqrt{(BN)^2+(AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{asin18^o}{\sqrt{(asin18^o)^2+(acos18^o)^2}}\right\} - \sin^{-1}\left\{\left(\frac{asin18^o}{\sqrt{(asin18^o)^2+(acos18^o)^2}}\right)\left(\frac{h}{\sqrt{h^2+(acos18^o)^2}}\right)\right\}$$

$$= \sin^{-1}\{sin18^o\} - \sin^{-1}\left\{(sin18^o)\left(\frac{h}{\sqrt{h^2+a^2cos^2 18^o}}\right)\right\}$$

$$\omega_{\Delta ANB} = \frac{\pi}{10} - \sin^{-1}\left(\frac{hsin18^o}{\sqrt{h^2+a^2cos^2 18^o}}\right)$$

Similarly, we get solid angle subtended by right ΔAQC at the given point 'P'

$$\omega_{\Delta AQC} = \sin^{-1}\left\{\frac{CQ}{\sqrt{(CQ)^2+(AQ)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CQ}{\sqrt{(CQ)^2+(AQ)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AQ)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}cot18^o\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{a}{2}cot18^o\right)^2}}\right)\left(\frac{h}{\sqrt{h^2+\left(\frac{a}{2}cot18^o\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1}{cosec18^o}\right\} - \sin^{-1}\left\{\left(\frac{1}{cosec18^o}\right)\left(\frac{2h}{\sqrt{4h^2+a^2cot^2 18^o}}\right)\right\}$$

$$= \frac{\pi}{10} - \sin^{-1}\left(\frac{2hsin18^o}{\sqrt{4h^2+a^2cot^2 18^o}}\right)$$

Hence on setting the corresponding values, solid angle subtended by the regular pentagonal plane ABCDE at the given point P is given as

$$\omega_{reg.penta.} = 2\left[\omega_{\Delta ANC} + \omega_{\Delta AQC} - \omega_{\Delta ANB}\right]$$

$$= 2\left[\frac{3\pi}{10} - \sin^{-1}\left(\frac{hcos36^o}{\sqrt{h^2 + a^2cos^2 18^o}}\right) + \frac{\pi}{10} - \sin^{-1}\left(\frac{2hsin18^o}{\sqrt{4h^2 + a^2cot^2 18^o}}\right) - \frac{\pi}{10} + \sin^{-1}\left(\frac{hsin18^o}{\sqrt{h^2 + a^2cos^2 18^o}}\right)\right]$$

$$\omega_{reg.penta.} = 2\left[\frac{3\pi}{10} + \sin^{-1}\left(\frac{hsin18^o}{\sqrt{h^2 + a^2cos^2 18^o}}\right) - \sin^{-1}\left(\frac{hcos36^o}{\sqrt{h^2 + a^2cos^2 18^o}}\right) - \sin^{-1}\left(\frac{2hsin18^o}{\sqrt{4h^2 + a^2cot^2 18^o}}\right)\right]$$

Note: This is the standard formula to find out the value of solid angle subtended by a regular pentagonal plane, having each side of length a, at any point lying at a normal height h from any of the vertices.

- **Regular Hexagonal Plane**

By following the same procedure as in case of a regular pentagon, we can divide the hexagon into sub-elementary right triangles.

(As shown in the figure 18)

By using Element Method, solid angle subtended by given regular hexagonal plane ABCDEF at the given point P lying at a normal height h from vertex 'A' (i.e. foot of perpendicular)

$$\Rightarrow \omega_{reg.hexa.} = 2(\omega_{\Delta ANC} + \omega_{\Delta ACD} - \omega_{\Delta ANB})$$

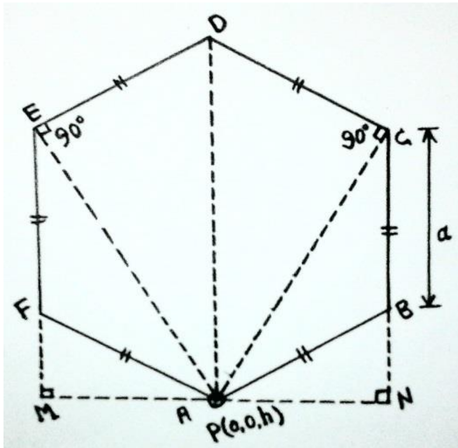


Fig 18: Point P lying on the normal axis passing through the vertex A of regular hexagon

Necessary dimensions can be calculated by the figure as follows

$$AN = acos30^o = \frac{a\sqrt{3}}{2},\ \ BN = asin30^o = \frac{a}{2},\ \ CN = a + \frac{a}{2} = \frac{3a}{2}\ \ \&\ AC = \frac{3a}{2sin60^o} = a\sqrt{3}$$

Now, using standard formula-1, solid angle subtended by the right ΔANC at the point $P(0, 0, h)$

$$\omega_{\Delta ANC} = \sin^{-1}\left\{\frac{CN}{\sqrt{(CN)^2 + (AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CN}{\sqrt{(CN)^2 + (AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\frac{3a}{2}}{\sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\frac{3a}{2}}{\sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{3}}{2}\right\} - \sin^{-1}\left\{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{2h}{\sqrt{4h^2 + 3a^2}}\right)\right\} = \frac{\pi}{3} - \sin^{-1}\left(\frac{h\sqrt{3}}{\sqrt{4h^2 + 3a^2}}\right)$$

Similarly, we get solid angle subtended by right ΔANB at the given point $P(0, 0, h)$

$$\omega_{\Delta ANB} = \sin^{-1}\left\{\frac{BN}{\sqrt{(BN)^2 + (AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BN}{\sqrt{(BN)^2 + (AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2 + (AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}}\right\} - \sin^{-1}\left\{\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2 + \left(\frac{a\sqrt{3}}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1}{2}\right\} - \sin^{-1}\left\{\left(\frac{1}{2}\right)\left(\frac{2h}{\sqrt{4h^2+3a^2}}\right)\right\} = \frac{\pi}{6} - \sin^{-1}\left(\frac{h}{\sqrt{4h^2+3a^2}}\right)$$

Similarly, we get solid angle subtended by right ΔACD at the given point $P(0,0,h)$

$$\omega_{\Delta ACD} = \sin^{-1}\left\{\frac{CD}{\sqrt{(CD)^2+(AC)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CD}{\sqrt{(CD)^2+(AC)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AC)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{a}{\sqrt{a^2+(a\sqrt{3})^2}}\right\} - \sin^{-1}\left\{\left(\frac{a}{\sqrt{h^2+(a\sqrt{3})^2}}\right)\left(\frac{h}{\sqrt{h^2+(a\sqrt{3})^2}}\right)\right\}$$

$$\omega_{\Delta ACD} = \sin^{-1}\left\{\frac{1}{2}\right\} - \sin^{-1}\left\{\left(\frac{1}{2}\right)\left(\frac{h}{\sqrt{h^2+3a^2}}\right)\right\} = \frac{\pi}{6} - \sin^{-1}\left(\frac{h}{2\sqrt{h^2+3a^2}}\right)$$

Hence on setting the corresponding values, solid angle subtended by the regular hexagonal plane ABCDEF at the given point P is given as

$$\omega_{reg.hexa.} = 2[\omega_{\Delta ANC} + \omega_{\Delta ACD} - \omega_{\Delta ANB}]$$

$$= 2\left[\frac{\pi}{3} - \sin^{-1}\left(\frac{h\sqrt{3}}{\sqrt{4h^2+3a^2}}\right) + \frac{\pi}{6} - \sin^{-1}\left(\frac{h}{2\sqrt{h^2+3a^2}}\right) - \frac{\pi}{6} + \sin^{-1}\left(\frac{h}{\sqrt{4h^2+3a^2}}\right)\right]$$

$$\boldsymbol{\omega_{reg.hexa.} = 2\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{h}{\sqrt{4h^2+3a^2}}\right) - \sin^{-1}\left(\frac{h\sqrt{3}}{\sqrt{4h^2+3a^2}}\right) - \sin^{-1}\left(\frac{h}{2\sqrt{h^2+3a^2}}\right)\right]}$$

Note: **This is the standard formula to find out the value of solid angle subtended by a regular hexagonal plane, having each side of length a, at any point lying at a normal height h from any of the vertices.**

Thus, all above standard results are obtained by analytical method of HCR's Theory using single **standard formula-1 of right triangular plane** only. It is obvious that this theory can be applied to find out the solid angle subtended by any polygonal plane (i.e. **plane bounded by the straight lines only**) provided the location of foot of perpendicular (**F.O.P**.) is known.

Now, we are interested to calculate solid angle subtended by different polygonal planes at different points in the space by tracing the diagram, specifying the F.O.P. & measuring the necessary dimensions & calculating.

X. Graphical Applications of Theory of Polygon

Graphical Method:

This method is similar to the analytical method which is applicable for some particular configurations of polygon & locations of given point in the space. But graphical method is applicable for any configuration of polygonal plane & location of the point in the space. This is the **method of tracing, measurements & mathematical calculations** which requires the following parameters to be already known

1. Geometrical shape & dimensions of the polygonal plane

2. Normal distance (h) of the given point from the plane of polygon

3. Location of foot of perpendicular (F.O.P.) drawn from given point to the plane of polygon

First let's know the working steps of the graphical method as follows

Step 1: Trace the diagram of the given polygon with the help of known sides & angles.

Step 2: Draw a perpendicular to the plane of polygon & specify the location of F.O.P.

Step 3: Divide the polygon into elementary triangles then each elementary triangle into two sub-elementary right triangles all having common vertex at the F.O.P.

Step 4: Find the area of the polygon as the algebraic sum of areas of sub-elementary right triangles i.e. area of each of the right triangles must be taken with proper sign (positive or negative depending on the area is inside or outside the boundary of polygon)

Step 5: Replace each area of sub-elementary right triangle by the solid angle subtended by that right triangle at the given point in the space.

Step 6: Measure the necessary dimensions (i.e. distances) & set them into standard formula-1 to calculate the solid angle subtended by each of the sub-elementary right triangles.

Step 7: Thus, find out the value of solid angle subtended by given polygonal plane at the given point by taking the algebraic sum of solid angles subtended by the sub-elementary right triangles at the same point in the space.

We are interested to directly apply the above steps without mentioning them in the following numerical examples

Numerical Examples:

Example 1: Let's find out the value of solid angle subtended by a triangular ABC having sides $AB = 8.6cm, BC = 4cm$ & $AC = 5.5cm$ at a point P lying at a normal height 3cm from the vertex 'A'.

Sol. Draw the triangle ABC with known values of the sides & specify the location of given point P by $P(0, 0, 3)$ perpendicularly outwards to the plane of paper & F.O.P. (i.e. vertex 'A') (as shown in the figure 19 below)

Divide ΔABC into two right triangles ΔANB & ΔACN by drawing a perpendicular AN to the opposite side BC (extended line). All must have common vertex at F.O.P.

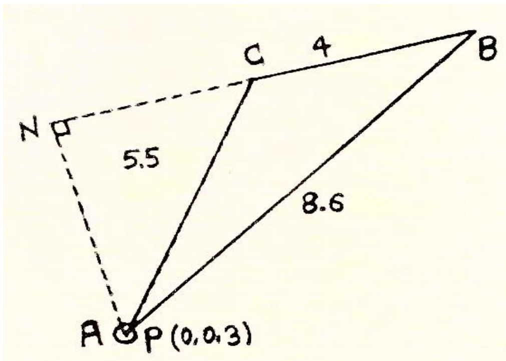


Fig 19: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

It is clear from the diagram, the solid angle subtended by ΔABC at the point 'P' is given by Element Method as follows

$$\omega_{\Delta ABC} = \omega_{\Delta ANB} - \omega_{\Delta ANC}$$

Now, measure the necessary dimensions & perform the following calculations, by using standard formula-1

$$AN = 4.4cm, \quad CN = 3.6cm \qquad (from\ the\ diagram)$$

$$BN = BC + CN = 4 + 3.6 = 7.6cm$$

Now, solid angle subtended by right ΔANB at the given point 'P'

On setting the corresponding values in formula of right triangle

$$\Rightarrow \boldsymbol{\omega_{\Delta ANB}} = \sin^{-1}\left\{\frac{BN}{\sqrt{(BN)^2+(AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BN}{\sqrt{(BN)^2+(AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{7.6}{\sqrt{7.6^2+4.4^2}}\right\} - \sin^{-1}\left\{\left(\frac{7.6}{\sqrt{7.6^2+4.4^2}}\right)\left(\frac{3}{\sqrt{3^2+4.4^2}}\right)\right\}$$

$$= 1.046000555 - 0.509254517 = \mathbf{0.536746038\ sr}$$

Similarly, solid angle subtended by right ΔANC at the given point 'P'

$$\Rightarrow \boldsymbol{\omega_{\Delta ANC}} = \sin^{-1}\left\{\frac{CN}{\sqrt{(CN)^2+(AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CN}{\sqrt{(CN)^2+(AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{3.6}{\sqrt{3.6^2+4.4^2}}\right\} - \sin^{-1}\left\{\left(\frac{3.6}{\sqrt{3.6^2+4.4^2}}\right)\left(\frac{3}{\sqrt{3^2+4.4^2}}\right)\right\}$$

$$= 0.68572951 - 0.364761147 = \mathbf{0.320968363\ sr}$$

Hence, solid angle subtended by ΔABC at the point 'P' (by Element Method)

$$\Rightarrow \quad \boldsymbol{\omega_{\Delta ABC}} = \omega_{\Delta ANB} - \omega_{\Delta ANC} = 0.536746038 - 0.320968363 = \mathbf{0.215777675\ sr}$$

Example 2: Let's find out solid angle subtended by a quadrilateral ABCD having sides $AB = 6cm, BC = 8cm$, $CD = 7cm, AD = 4cm \,\&\, \angle BAD = 110^o$ at a point lying at a normal height 2cm from the vertex 'A' & calculate the total luminous flux intercepted by the plane ABCD if a uniform point-source of 1400 lm is located at the point 'P'

Sol. Draw the quadrilateral ABCD with known values of the sides & angle & specify the location of given point P by $P(0,0,2)$ perpendicularly outwards to the plane of paper & F.O.P. (i.e. vertex 'A') (as shown in the fig 20)

Divide the quadrilateral ABCD into two triangles $\Delta ABC \,\&\, \Delta ADC$ by joining the vertex C to the F.O.P. 'A'. Further divide $\Delta ABC \,\&\, \Delta ADC$ into two right triangles $\Delta AMB \,\&\, \Delta AMC$ and $\Delta ANC \,\&\, \Delta AND$ respectively simply by drawing perpendicular to the opposite side in ΔABC & ΔADC having common vertex at F.O.P.

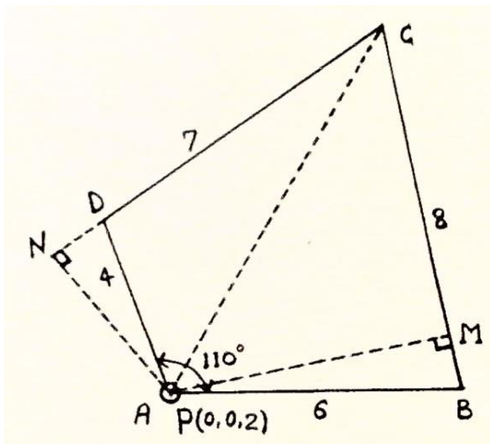


Fig 20: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

It is clear from the diagram, the solid angle subtended by quadrilateral ABCD at the point 'P' is given by Element Method

$$\omega_{ABCD} = \omega_{\Delta ABC} + \omega_{\Delta ACD} = (\omega_{\Delta AMC} + \omega_{\Delta AMB}) + (\omega_{\Delta ANC} - \omega_{\Delta AND})$$

Now, measure the necessary dimensions then perform the following calculations

$AN = 3.8cm,\ \ DN = 1.2cm,$
$AM = 5.9cm, \qquad (from\ the\ diagram)$

$BM = 1.2cm \Rightarrow CN = CD + DN = 7 + 1.2 = 8.2cm$ &

$CM = BC - BM = 8 - 1.2 = 6.8cm$

Now, solid angle subtended by right ΔANC at the given point 'P'

On setting the corresponding values in formula of right triangle

$$\Rightarrow \boldsymbol{\omega_{\Delta ANC}} = \sin^{-1}\left\{\frac{CN}{\sqrt{(CN)^2+(AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CN}{\sqrt{(CN)^2+(AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{8.2}{\sqrt{8.2^2+3.8^2}}\right\} - \sin^{-1}\left\{\left(\frac{8.2}{\sqrt{8.2^2+3.8^2}}\right)\left(\frac{2}{\sqrt{2^2+3.8^2}}\right)\right\}$$

$$= 1.136842957 - 0.436286431 = \boldsymbol{0.700556526\ sr}$$

Similarly, solid angle subtended by right ΔAND at the given point 'P'

$$\Rightarrow \boldsymbol{\omega_{\Delta AND}} = \sin^{-1}\left\{\frac{DN}{\sqrt{(DN)^2+(AN)^2}}\right\} - \sin^{-1}\left\{\left(\frac{DN}{\sqrt{(DN)^2+(AN)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AN)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1.2}{\sqrt{1.2^2+3.8^2}}\right\} - \sin^{-1}\left\{\left(\frac{1.2}{\sqrt{1.2^2+3.8^2}}\right)\left(\frac{2}{\sqrt{2^2+3.8^2}}\right)\right\}$$

$$= 0.305878871 - 0.140714774 = \boldsymbol{0.165164097\ sr}$$

Similarly, solid angle subtended by right ΔAMC at the given point 'P'

$$\Rightarrow \boldsymbol{\omega_{\Delta AMC}} = \sin^{-1}\left\{\frac{CM}{\sqrt{(CM)^2+(AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CM}{\sqrt{(CM)^2+(AM)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AM)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{6.8}{\sqrt{6.8^2+5.9^2}}\right\} - \sin^{-1}\left\{\left(\frac{6.8}{\sqrt{6.8^2+5.9^2}}\right)\left(\frac{2}{\sqrt{2^2+5.9^2}}\right)\right\}$$

$$= 0.856146031 - 0.24492976 = \boldsymbol{0.611216271\ sr}$$

Similarly, solid angle subtended by right ΔAMB at the given point 'P'

$$\Rightarrow \boldsymbol{\omega_{\Delta AMB}} = \sin^{-1}\left\{\frac{BM}{\sqrt{(BM)^2+(AM)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BM}{\sqrt{(BM)^2+(AM)^2}}\right)\left(\frac{PA}{\sqrt{(PA)^2+(AM)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1.2}{\sqrt{1.2^2+5.9^2}}\right\} - \sin^{-1}\left\{\left(\frac{1.2}{\sqrt{1.2^2+5.9^2}}\right)\left(\frac{2}{\sqrt{2^2+5.9^2}}\right)\right\}$$

$$= 0.200652877 - 0.064029809 = \boldsymbol{0.136623068\ sr}$$

Hence, solid angle subtended by $quadrilateral\ ABCD$ at the point 'P' (by Element Method)

$$\Rightarrow \quad \omega_{ABCD} = \omega_{\Delta ABC} + \omega_{\Delta ACD} = (\omega_{\Delta AMC} + \omega_{\Delta AMB}) + (\omega_{\Delta ANC} - \omega_{\Delta AND})$$

$$\therefore \quad \boldsymbol{\omega_{ABCD}} = 0.611216271 + 0.136623068 + 0.700556526 - 0.165164097 = \boldsymbol{1.283231768\ sr}$$

Calculation of Luminous Flux: If a **uniform point-source** of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the quadrilateral plane ABCD

$$= \frac{solid\ angle \times total\ flux\ emitted\ by\ source}{4\pi} = \frac{1.283231768 \times 1400}{4\pi} = \boldsymbol{142.9628753\ lm\ (Lumen)}$$

It means that only 142.9628753 lm out of 1400 lm flux is striking the quadrilateral plane ABCD & rest of the flux is escaping to the surrounding space. This result can be experimentally verified. (H.C. Rajpoot)

Example 3: Let's find out solid angle subtended by a pentagonal plane ABCDE having sides $AB = 6cm, BC = 5.7cm,\ CD = 6.65cm, DE = 5.5cm, AE = 7.8cm,\ \angle BAE = 120^o\ \&\ \angle ABC = 80^o$ at a point 'P' lying at a normal height 6cm from a point 'O' internally dividing the side AB such that $OA{:}\,OB = 2{:}\,1$ & calculate the total luminous flux intercepted by the plane ABCDE if a uniform point-source of 1400 lm is located at the point 'P'

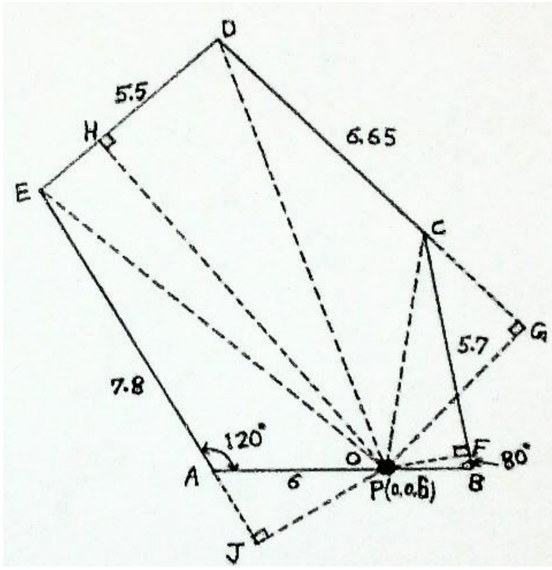


Fig 21: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

Sol: Draw the pentagon ABCDE with known values of the sides & angles & specify the location of given point P by $P(0, 0, 6)$ perpendicularly outwards to the plane of paper & F.O.P. 'O' (as shown in the figure 21)

Divide the pentagon ABCDE into elementary triangles $\Delta OBC, \Delta OCD, \Delta ODE\ \&\ \Delta OEA$ by joining all the vertices of pentagon ABCDE to the F.O.P. 'O'. Further divide each of the triangles $\Delta OBC, \Delta OCD, \Delta ODE\ \&\ \Delta OEA$ in two right triangles simply by drawing a perpendicular to the opposite side in the respective triangle. (See the diagram)

It is clear from the diagram, the solid angle subtended by pentagonal plane ABCDE at the point 'P' is given by Element Method as follows

$$\omega_{ABCDE} = \omega_{\Delta OBC} + \omega_{\Delta OCD} + \omega_{\Delta ODE} + \omega_{\Delta OEA} \quad \dots\dots\dots (I)$$

From the diagram, it's obvious that the solid angle ω_{ABCDE} subtended by the pentagon ABCDE is expressed as the algebraic sum of solid angles of sub-elementary right triangles only as follows

$$\omega_{\Delta OBC} = \omega_{\Delta OFB} + \omega_{\Delta OFC} \qquad \omega_{\Delta OCD} = \omega_{\Delta OGD} - \omega_{\Delta OGC}$$
$$\omega_{\Delta ODE} = \omega_{\Delta OHD} + \omega_{\Delta OHE} \qquad \omega_{\Delta OEA} = \omega_{\Delta OJE} - \omega_{\Delta OJA}$$

Now, setting the values in eq(I), we get

$$\omega_{ABCDE} = (\omega_{\Delta OFB} + \omega_{\Delta OFC}) + (\omega_{\Delta OGD} - \omega_{\Delta OGC}) + (\omega_{\Delta OHD} + \omega_{\Delta OHE}) + (\omega_{\Delta OJE} - \omega_{\Delta OJA}) \quad \dots\dots\dots\dots\dots (II)$$

Now, measure the necessary dimensions & set them into standard formula-1 to find out above values of solid angle subtended by the sub-elementary right triangles at the given point 'P' as follows

$$\Rightarrow \boldsymbol{\omega_{\Delta OFB}} = \sin^{-1}\left\{\frac{FB}{\sqrt{(FB)^2 + (OF)^2}}\right\} - \sin^{-1}\left\{\left(\frac{FB}{\sqrt{(FB)^2 + (OF)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{0.35}{\sqrt{(0.35)^2 + (1.95)^2}}\right\} - \sin^{-1}\left\{\left(\frac{0.35}{\sqrt{(0.35)^2 + (1.95)^2}}\right)\left(\frac{6}{\sqrt{(6)^2 + (1.95)^2}}\right)\right\}$$

$$= 0.177596167 - 0.168814218 = \mathbf{0.008781949}\ \boldsymbol{sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OFC}} = \sin^{-1}\left\{\frac{FC}{\sqrt{(FC)^2 + (OF)^2}}\right\} - \sin^{-1}\left\{\left(\frac{FC}{\sqrt{(FC)^2 + (OF)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{5.35}{\sqrt{(5.35)^2 + (1.95)^2}}\right\} - \sin^{-1}\left\{\left(\frac{5.35}{\sqrt{(5.35)^2 + (1.95)^2}}\right)\left(\frac{6}{\sqrt{(6)^2 + (1.95)^2}}\right)\right\}$$

$$= 1.221275136 - 1.105149872 = \mathbf{0.116125264}\ \boldsymbol{sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OGD}} = \sin^{-1}\left\{\frac{DG}{\sqrt{(DG)^2+(OG)^2}}\right\} - \sin^{-1}\left\{\left(\frac{DG}{\sqrt{(DG)^2+(OG)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OG)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{9.9}{\sqrt{(9.9)^2+(4.6)^2}}\right\} - \sin^{-1}\left\{\left(\frac{9.9}{\sqrt{(9.9)^2+(4.6)^2}}\right)\left(\frac{6}{\sqrt{(6)^2+(4.6)^2}}\right)\right\}$$

$$= 1.135829376 - 0.803382922 = \mathbf{0.332446454}\ \boldsymbol{sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OGC}} = \sin^{-1}\left\{\frac{CG}{\sqrt{(CG)^2+(OG)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CG}{\sqrt{(CG)^2+(OG)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OG)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{3.25}{\sqrt{(3.25)^2+(4.6)^2}}\right\} - \sin^{-1}\left\{\left(\frac{3.25}{\sqrt{(3.25)^2+(4.6)^2}}\right)\left(\frac{6}{\sqrt{(6)^2+(4.6)^2}}\right)\right\}$$

$$= 0.615089573 - 0.475672072 = \mathbf{0.139417501}\ \boldsymbol{sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OHD}} = \sin^{-1}\left\{\frac{DH}{\sqrt{(DH)^2+(OH)^2}}\right\} - \sin^{-1}\left\{\left(\frac{DH}{\sqrt{(DH)^2+(OH)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OH)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{3.7}{\sqrt{(3.7)^2+(10.25)^2}}\right\} - \sin^{-1}\left\{\left(\frac{3.7}{\sqrt{(3.7)^2+(10.25)^2}}\right)\left(\frac{6}{\sqrt{(6)^2+(10.25)^2}}\right)\right\}$$

$$= 0.346418989 - 0.172376751 = \mathbf{0.174042238}\ \boldsymbol{sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OHE}} = \sin^{-1}\left\{\frac{EH}{\sqrt{(EH)^2+(OH)^2}}\right\} - \sin^{-1}\left\{\left(\frac{EH}{\sqrt{(EH)^2+(OH)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OH)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1.8}{\sqrt{(1.8)^2+(10.25)^2}}\right\} - \sin^{-1}\left\{\left(\frac{1.8}{\sqrt{(1.8)^2+(10.25)^2}}\right)\left(\frac{6}{\sqrt{(6)^2+(10.25)^2}}\right)\right\}$$

$$= 0.173837242 - 0.087488889 = \mathbf{0.086348353}\ \boldsymbol{sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OJE}} = \sin^{-1}\left\{\frac{EJ}{\sqrt{(EJ)^2+(OJ)^2}}\right\} - \sin^{-1}\left\{\left(\frac{EJ}{\sqrt{(EJ)^2+(OJ)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OJ)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{9.8}{\sqrt{(9.8)^2+(3.45)^2}}\right\} - \sin^{-1}\left\{\left(\frac{9.8}{\sqrt{(9.8)^2+(3.45)^2}}\right)\left(\frac{6}{\sqrt{(6)^2+(3.45)^2}}\right)\right\}$$

$$= 1.232304566 - 0.957430258 = \mathbf{0.274874308}\ \boldsymbol{sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OJA}} = \sin^{-1}\left\{\frac{AJ}{\sqrt{(AJ)^2+(OJ)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AJ}{\sqrt{(AJ)^2+(OJ)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OJ)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{2}{\sqrt{(2)^2+(3.45)^2}}\right\} - \sin^{-1}\left\{\left(\frac{2}{\sqrt{(2)^2+(3.45)^2}}\right)\left(\frac{6}{\sqrt{(6)^2+(3.45)^2}}\right)\right\}$$

$$= 0.525366873 - 0.449793847 = \mathbf{0.075573026}\ \boldsymbol{sr}$$

Hence, by setting the corresponding values in eq(II), solid angle subtended by the pentagonal plane at the given point 'P' is calculated as follows

$$\boldsymbol{\omega_{ABCDE}} = (\omega_{\Delta OFB} + \omega_{\Delta OFC}) + (\omega_{\Delta OGD} - \omega_{\Delta OGC}) + (\omega_{\Delta OHD} + \omega_{\Delta OHE}) + (\omega_{\Delta OJE} - \omega_{\Delta OJA})$$

$$= (0.008781949 + 0.116125264) + (0.332446454 - 0.139417501) + (0.174042238 + 0.086348353) + (0.274874308 - 0.075573026) = \mathbf{0.777628039\ sr}$$

Calculation of Luminous Flux: If a uniform point-source of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the pentagonal plane ABCDE

$$= \frac{solid\ angle \times total\ flux\ emitted\ by\ source}{4\pi} = \frac{0.777628039 \times 1400}{4\pi} = \mathbf{86.63434241\ lm\ (Lumen)}$$

It means that only 86.63434241 lm out of 1400 lm flux is striking the pentagonal plane ABCDE & rest of the flux is escaping to the surrounding space. This result can be experimentally verified. (H.C. Rajpoot)

Example 4: Let's find out solid angle subtended by a quadrilateral plane ABCD having sides $AB = 8cm, BC = 9cm,\ CD = 6cm, AD = 4cm\ \&\ \angle BAD = 70^o$ at a point 'P' lying at a normal height 4cm from a point 'O' outside the quadrilateral ABCD such that $OB = 8cm\ \&\ OC = 4.2cm$ & calculate the total luminous flux intercepted by the plane ABCD if a uniform point-source of 1400 lm is located at the point 'P'

Sol: Draw the quadrilateral ABCD with known values of the sides & angle & specify the location of given point P by $P(0, 0, 4)$ perpendicularly outwards to the plane of paper & F.O.P. 'O' (See the figure 22)

Divide quadrilateral ABCD into elementary triangles $\Delta OAB, \Delta ODA\ \&\ \Delta OCD$ by joining all the vertices of quadrilateral ABCD to the F.O.P. 'O'. Further divide each of the triangles $\Delta OAB, \Delta ODA\ \&\ \Delta OCD$ in two right triangles simply by drawing perpendiculars OE, OG & OF to the opposite sides AB, AD & CD in the respective triangles. (See the diagram)

It is clear from the diagram, the solid angle subtended by quadrilateral plane ABCD at the given point 'P' is given by **Element Method** as follows

Area of quadrilateral ABCD = algebraic sum of areas of elementary triangles

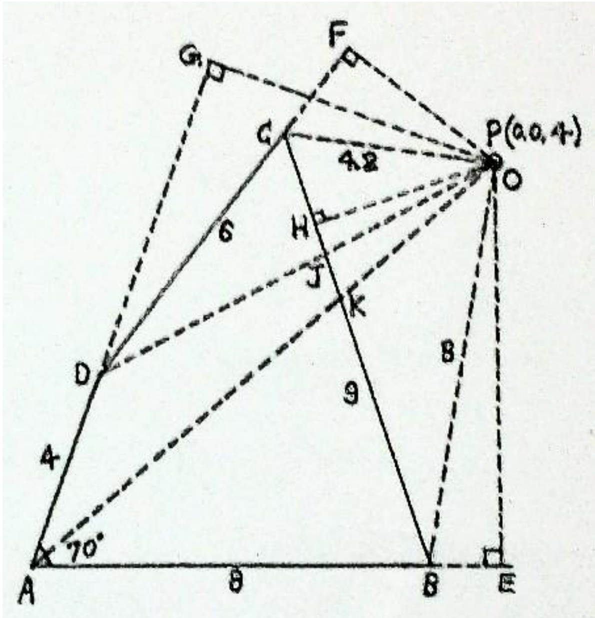


Fig 22: Point P is lying perpendicularly outwards to the plane of paper. All the dimensions are in cm.

$$\therefore\ A_{ABCD} = (A_{\Delta OAB} - A_{\Delta OKB}) + (A_{\Delta ODA} - A_{\Delta OJK}) + (A_{\Delta OCD} - A_{\Delta OCJ})$$

Now, replacing areas by corresponding values of solid angle, we get

$$\omega_{ABCD} = (\omega_{\Delta OAB} - \omega_{\Delta OKB}) + (\omega_{\Delta ODA} - \omega_{\Delta OJK}) + (\omega_{\Delta OCD} - \omega_{\Delta OCJ}) \quad \dots\dots\dots (I)$$

Now, draw a perpendicular OH from F.O.P. to the side BC to divide $\Delta OKB, \Delta OJK\ \&\ \Delta OCJ$ into right triangles & express the above values of solid angle as the algebraic sum of solid angles subtended by the right triangles only as follows

$$\omega_{\Delta OAB} = \omega_{\Delta OEA} - \omega_{\Delta OEB} \qquad \omega_{\Delta ODA} = \omega_{\Delta OGA} - \omega_{\Delta OGD} \qquad \omega_{\Delta OCD} = \omega_{\Delta OFD} - \omega_{\Delta OFC}$$
$$\omega_{\Delta OKB} = \omega_{\Delta OHB} - \omega_{\Delta OHK} \qquad \omega_{\Delta OJK} = \omega_{\Delta OHK} - \omega_{\Delta OHJ} \qquad \omega_{\Delta OCJ} = \omega_{\Delta OHC} + \omega_{\Delta OHJ}$$

Now, setting the above values in eq(I), we get

$$\omega_{ABCD} = (\omega_{\Delta OEA} - \omega_{\Delta OEB} - \omega_{\Delta OHB} + \omega_{\Delta OHK}) + (\omega_{\Delta OGA} - \omega_{\Delta OGD} - \omega_{\Delta OHK} + \omega_{\Delta OHJ}) + (\omega_{\Delta OFD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} - \omega_{\Delta OHJ})$$

$$\omega_{ABCD} = \omega_{\Delta OEA} + \omega_{\Delta OGA} + \omega_{\Delta OFD} - \omega_{\Delta OEB} - \omega_{\Delta OHB} - \omega_{\Delta OGD} - \omega_{\Delta OFC} - \omega_{\Delta OHC} \quad \dots\dots\dots..(II)$$

Now, measure the necessary dimensions & set them into standard formula-1 to find out above values of solid angle subtended by the sub-elementary right triangles at the given point 'P' as follows

$$\Rightarrow \boldsymbol{\omega_{\Delta OEA}} = \sin^{-1}\left\{\frac{AE}{\sqrt{(AE)^2+(OE)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AE}{\sqrt{(AE)^2+(OE)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OE)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{9.4}{\sqrt{(9.4)^2+(7.9)^2}}\right\} - \sin^{-1}\left\{\left(\frac{9.4}{\sqrt{(9.4)^2+(7.9)^2}}\right)\left(\frac{4}{\sqrt{(4)^2+(7.9)^2}}\right)\right\}$$

$$= 0.871887063 - 0.353107934 = \mathbf{0.518779129\ sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OGA}} = \sin^{-1}\left\{\frac{AG}{\sqrt{(AG)^2+(OG)^2}}\right\} - \sin^{-1}\left\{\left(\frac{AG}{\sqrt{(AG)^2+(OG)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OG)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{10.5}{\sqrt{(10.5)^2+(6)^2}}\right\} - \sin^{-1}\left\{\left(\frac{10.5}{\sqrt{(10.5)^2+(6)^2}}\right)\left(\frac{4}{\sqrt{(4)^2+(6)^2}}\right)\right\}$$

$$= 1.051650213 - 0.502496173 = \mathbf{0.54915404\ sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OFD}} = \sin^{-1}\left\{\frac{DF}{\sqrt{(DF)^2+(OF)^2}}\right\} - \sin^{-1}\left\{\left(\frac{DF}{\sqrt{(DF)^2+(OF)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OF)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{8.1}{\sqrt{(8.1)^2+(3.65)^2}}\right\} - \sin^{-1}\left\{\left(\frac{8.1}{\sqrt{(8.1)^2+(3.65)^2}}\right)\left(\frac{4}{\sqrt{(4)^2+(3.65)^2}}\right)\right\}$$

$$= 1.147429185 - 0.738889475 = \mathbf{0.408539709\ sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OEB}} = \sin^{-1}\left\{\frac{BE}{\sqrt{(BE)^2+(OE)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BE}{\sqrt{(BE)^2+(OE)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OE)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1.4}{\sqrt{(1.4)^2+(7.9)^2}}\right\} - \sin^{-1}\left\{\left(\frac{1.4}{\sqrt{(1.4)^2+(7.9)^2}}\right)\left(\frac{4}{\sqrt{(4)^2+(7.9)^2}}\right)\right\}$$

$$= 0.17539422 - 0.078906232 = \mathbf{0.096487988\ sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OHB}} = \sin^{-1}\left\{\frac{BH}{\sqrt{(BH)^2+(OH)^2}}\right\} - \sin^{-1}\left\{\left(\frac{BH}{\sqrt{(BH)^2+(OH)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OH)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{7.1}{\sqrt{(7.1)^2+(3.8)^2}}\right\} - \sin^{-1}\left\{\left(\frac{7.1}{\sqrt{(7.1)^2+(3.8)^2}}\right)\left(\frac{4}{\sqrt{(4)^2+(3.8)^2}}\right)\right\}$$

$$= 1.079378081 - 0.69346571 = \mathbf{0.385912371\ sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OGD}} = \sin^{-1}\left\{\frac{DG}{\sqrt{(DG)^2+(OG)^2}}\right\} - \sin^{-1}\left\{\left(\frac{DG}{\sqrt{(DG)^2+(OG)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2+(OG)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{6.5}{\sqrt{(6.5)^2+(6)^2}}\right\} - \sin^{-1}\left\{\left(\frac{6.5}{\sqrt{(6.5)^2+(6)^2}}\right)\left(\frac{4}{\sqrt{(4)^2+(6)^2}}\right)\right\}$$

$$= 0.82537685 - 0.419819479 = \mathbf{0.405557371\ sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OFC}} = \sin^{-1}\left\{\frac{CF}{\sqrt{(CF)^2 + (OF)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CF}{\sqrt{(CF)^2 + (OF)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2 + (OF)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{2.1}{\sqrt{(2.1)^2 + (3.65)^2}}\right\} - \sin^{-1}\left\{\left(\frac{2.1}{\sqrt{(2.1)^2 + (3.65)^2}}\right)\left(\frac{4}{\sqrt{(4)^2 + (3.65)^2}}\right)\right\}$$

$$= 0.522091613 - 0.37726383 = \mathbf{0.144827783\ sr}$$

$$\Rightarrow \boldsymbol{\omega_{\Delta OHC}} = \sin^{-1}\left\{\frac{CH}{\sqrt{(CH)^2 + (OH)^2}}\right\} - \sin^{-1}\left\{\left(\frac{CH}{\sqrt{(CH)^2 + (OH)^2}}\right)\left(\frac{PO}{\sqrt{(PO)^2 + (OH)^2}}\right)\right\}$$

$$= \sin^{-1}\left\{\frac{1.9}{\sqrt{(1.9)^2 + (3.8)^2}}\right\} - \sin^{-1}\left\{\left(\frac{1.9}{\sqrt{(1.9)^2 + (3.8)^2}}\right)\left(\frac{4}{\sqrt{(4)^2 + (3.8)^2}}\right)\right\}$$

$$= 0.463647609 - 0.330197223 = \mathbf{0.133450386\ sr}$$

Hence, by setting the corresponding values in eq(II), solid angle subtended by the pentagonal plane at the given point 'P' is calculated as follows

$$\boldsymbol{\omega_{ABCD}} = \omega_{\Delta OEA} + \omega_{\Delta OGA} + \omega_{\Delta OFD} - \omega_{\Delta OEB} - \omega_{\Delta OHB} - \omega_{\Delta OGD} - \omega_{\Delta OFC} - \omega_{\Delta OHC}$$

$$= 0.518779129 + 0.54915404 + 0.408539709 - 0.096487988 - 0.385912371 - 0.405557371 - 0.144827783 - 0.133450386 = \mathbf{0.310236979\ sr}$$

Calculation of Luminous Flux: If a **uniform point-source** of 1400 lm is located at the given point 'P' then the **total luminous flux intercepted** by the quadrilateral plane ABCD

$$= \frac{solid\ angle \times total\ flux\ emitted\ by\ source}{4\pi} = \frac{0.310236979 \times 1400}{4\pi} = \mathbf{34.56302412\ lm\ (Lumen)}$$

It means that only 34.56302412 lm out of 1400 lm flux is striking the quadrilateral plane ABCD & rest of the flux is escaping to the surrounding space. This result can be experimentally verified.

Thus, all the mathematical results obtained above can be verified by the experimental results. Although, there had not been any unifying principle to be applied on any polygonal plane for any configuration & location of the point in the space. The symbols & names used above are arbitrary given by the author Mr H.C. Rajpoot.

XI. Conclusion

It is obvious from results obtained above that this theory is a **Unifying Principle** which is easy to apply for any configuration of a given polygon & any location of a point (i.e. observer) in the space by using a simple & systematic procedure & a standard formula. Necessary dimensions can be measured by analytical method or by tracing the diagram of polygon & specifying the location of F.O.P.
Though, it is a little lengthy for random configuration of polygon & location of observer still it can be applied to find the solid angle subtended by polygon in the easier way as compared to any other methods existing so far in the field of 3-D Geometry. Theory of Polygon can be concluded as follows

Applicability: It is easily applied to find out the solid angle subtended by any polygonal plane (i.e. **plane bounded by the straight lines only**) at any point (i.e. observer) in 3D space.

Conditions of Application: This theory is applicable for any polygonal plane & any point in the space if the following parameters are already known

1. Geometrical shape & dimensions of the polygonal plane

2. Normal distance (h) of the given point from the plane of polygon

3. Location of foot of perpendicular (F.O.P.) drawn from given point to the plane of polygon

While, the necessary dimensions (values) used in **master formula-1** (as derived above) are calculated either by analytical method or by graphical method i.e. by tracing the diagram & measuring the dimensions depending on which is easier. Analytical method is limited for some particular location of the point while Graphical method is applicable for all the locations & configuration of polygon w.r.t. the observer in the space. This method can never fail but a little complexity may be there in case of random locations & polygon with higher number of sides.

Steps to be followed:

1. Trace the diagram of the given polygon with the help of known sides & angles.

2. Draw a perpendicular to the plane of polygon & specify the location of F.O.P.

3. Divide the polygon into right triangles having common vertex at the F.O.P. & find the solid angle subtended by the polygon as the algebraic sum of solid angles subtended by right triangles such that algebraic sum of areas of right triangles is equal to the area of polygon.

4. Measure the necessary dimensions & set them into standard formula-1 to calculate solid angle subtended by each of the right triangles & hence solid angle subtended by the polygon at the given point.

Ultimate aim is to find out solid angle, subtended by a polygon at a given point, as the algebraic sum of solid angles subtended by the right triangles, measuring the dimensions, applying **Master/standard formula-1** on each of the right triangle & calculating the required result.

Future Scope: This theory can be easily applied for finding out the solid angle subtended by **3-D objects** which have **surface bounded by the planes only** Ex. **Cube**, **Cuboid**, **Prism**, **Pyramid**, **Tetrahedron** etc. in 3-D modelling & analysis by tracing the profile of surface of the solid as a polygon in 2-D & specifying the location of a given point & F.O.P. in the plane of profile-polygon as the projection of such solids in 2-D is always a polygon for any configuration of surface of solid in 3D space.

Note: This Theory had been proposed by the author **Mr Harish Chandra Rajpoot (B Tech, ME)**

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