$Mathematical\ Analysis\ of\ Regular\ \ Spherical\ Polygons$

$Application\ of\ HCR's\ Theory\ of\ Polygon$

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1. Introduction: We very well know that a **regular spherical polygon is a regular polygon drawn on the surface of a sphere having certain radius.** It mainly differs from a regular plane polygon by having each of its sides as an arc of the great circle. Each of its sides is of equal length & each of its interior angles is of equal magnitude greater than the interior angle of a regular plane polygon having equal no. of sides. But the plane angle $\left(= {}^{2\pi}/_{n}\right)$ subtended by each of the sides of a regular spherical polygon at its centre is equal to that subtended by each of the sides of a regular plane polygon with the same no. of sides. (See figure 1)

2. Analysis of regular spherical polygon: Consider a regular polygon $A_1A_2A_3 \ldots \ldots . A_{n-1}A_n$ having n no. of sides each of length a & each interior angle θ drawn on a sphere having a radius R.

Relation of the important parameters $\boldsymbol{R}$, $\boldsymbol{n}, \boldsymbol{\theta}$ & $\boldsymbol{a}$: Let's derive a simple mathematical relation among four important parameters R, n, θ & a for any regular spherical polygon in order to calculate all **its important parameters as solid angle, area** etc.

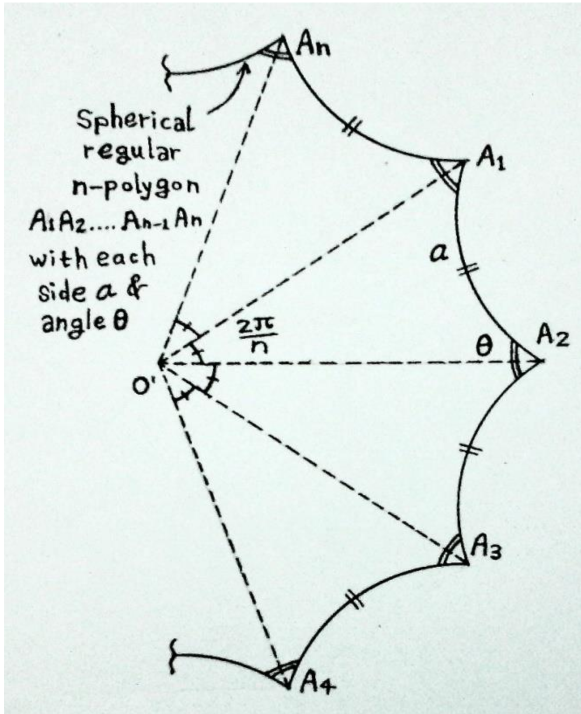

Figure 1: a regular spherical polygon $A_1A_2A_3 \ldots \ldots . A_{n-1}A_n$ having n no. of sides each as a great circle arc of length a, each interior angle θ & centre at the point O'

$\boldsymbol{R} = radius\ of\ the\ sphere$

$\boldsymbol{n} = no.of\ sides\ of\ regular\ spherical\ polygon$

$$(\forall\ \ n \in N\ \&\ \ n \geq 3)$$

$$\boldsymbol{\theta} =\ interior\ angle \qquad \left(\forall\ \ \frac{(n-2)\pi}{n} < \theta < \pi\right)$$

$\boldsymbol{a} = length\ of\ each\ side\ of\ regular\ spherical\ polygon$

$\boldsymbol{\alpha} = parametric\ angle\ of\ \ regular\ spherical\ polygon = {}^{a}/_{R}$

$\boldsymbol{\omega} = solid\ angle\ subtended\ by\ regular\ polygon\ at\ \ the\ \ centre\ of\ sphere$

$\boldsymbol{A} = area\ of\ regular\ polygon\ on\ the\ surface\ of\ sphere$

$\boldsymbol{a'} =$
$length\ of\ each\ side\ of\ corresponding\ regular\ plane\ polygon\ obtained\ by\ consecutively\ joining$

$all\ the\ vertices\ of\ a\ regular\ spherical\ polygon\ by\ straight\ lines$

$\boldsymbol{A'} = area\ of\ corresponding\ \ regular\ plane\ polygon$

Let's consider two consecutive sides A_1A_2 & A_1A_n on great circle arcs with common vertex A_1 & draw two tangents at the common vertex A_1 which intersect the extended lines, drawn from the centre point O passing through the vertices A_2 & A_n, at the points B & C respectively thus we obtain two tangents A_1B & A_1C at the vertex A_1 which make an angle equal to the interior angle θ with each other (See figure 2 below)

$$\text{Angle subtended by each side of regular polygon at the centre of sphere} = \frac{arc\ length}{radius} = \frac{A_1A_2}{OA_1} = \frac{a}{R}$$

$$\Rightarrow \boldsymbol{parametric\ angle,\ \alpha = \frac{a}{R}} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(I)$$

Now, in right ΔOA_1B

$$\Rightarrow \tan\angle A_1OB = \frac{A_1B}{OA_1} \Rightarrow tan\alpha = \frac{A_1B}{R} \Rightarrow \boldsymbol{A_1B = Rtan\alpha} \quad \&$$

$$\Rightarrow \cos\angle A_1OB = \frac{OA_1}{OB} \Rightarrow cos\alpha = \frac{R}{OB} \Rightarrow \boldsymbol{OB = \frac{R}{cos\alpha}}$$

Now join the points B & C & draw a perpendicular from the vertex A_1 to the line (side) BC at the point N. Join the point N to the centre O of the sphere. (See figure 2 below)

Now in rightΔA_1NB

$$\Rightarrow \sin\angle BA_1N = \frac{BN}{A_1B} \Rightarrow sin\frac{\theta}{2} = \frac{BN}{Rtan\alpha}$$

$$\Rightarrow \boldsymbol{BN = Rtan\alpha sin\frac{\theta}{2}}$$

In rightΔONB

$$\Rightarrow \sin\angle BON = \frac{BN}{OB} \Rightarrow sin\beta = \frac{Rtan\alpha sin\frac{\theta}{2}}{\left(\frac{R}{cos\alpha}\right)}$$

$$\boldsymbol{sin\beta = sin\alpha sin\frac{\theta}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots.(II)$$

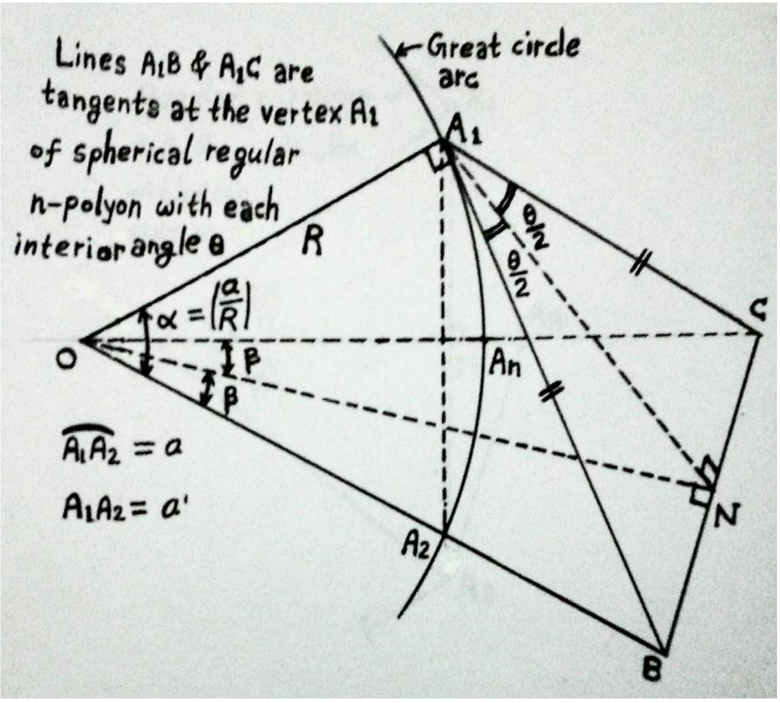

Figure 2: A regular polygon, with n no. of sides each of length $\boldsymbol{a}$ (measured along great circle arc) & each interior angle $\boldsymbol{\theta}$, is drawn on a sphere of radius $\boldsymbol{R}$. Two tangents $\boldsymbol{A_1B}$ & $\boldsymbol{A_1C}$ are drawn at the vertex $\boldsymbol{A_1}$ which make an angle equal to the interior angle $\boldsymbol{\theta < \pi}$ with each other.

Now, join all the vertices of regular spherical polygon by straight lines to obtain a regular plane polygon $A_1A_2A_3\ldots\ldots.A_{n-1}A_n$ having n no. of sides each of equal length a' (See figure 3 below) side $A_1A_2 = a'$ is obtained as follows

In isosceles ΔOA_1A_2

$$\Rightarrow \sin\frac{\angle A_1OA_2}{2} = \frac{\left(\frac{A_1A_2}{2}\right)}{OA_1} \Rightarrow sin\frac{\alpha}{2} = \frac{\left(\frac{a'}{2}\right)}{R} \Rightarrow \boldsymbol{a' = 2Rsin\frac{\alpha}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(III)$$

Now, in regular plane polygon, join the vertices A_1, A_2 & A_3 to the centre O' & the line joining the vertices A_1& A_3 normally intersects $O'A_2$ at the point M. (See figure 3 below)

$$\text{circumscribed radius of regular plane polygon} = O'A_1 = O'A_2 = \ldots\ldots\ldots.= O'A_n = \frac{A_1A_2}{2sin\frac{\pi}{n}}$$

$$\Rightarrow \boldsymbol{O'A_1 = \frac{a'}{2sin\frac{\pi}{n}} = \frac{2Rsin\frac{\alpha}{2}}{2sin\frac{\pi}{n}} = \frac{Rsin\frac{\alpha}{2}}{sin\frac{\pi}{n}}}$$

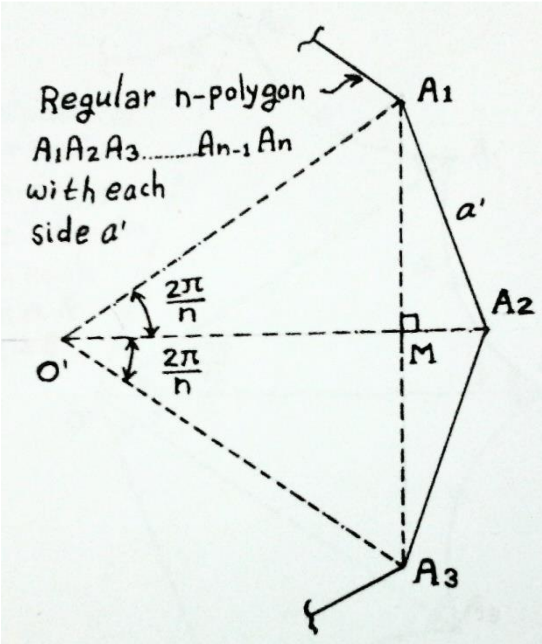

Figure 3: A regular plane polygon $A_1A_2A_3\ldots\ldots A_{n-1}A_n$,having n no. of sides each of length $\boldsymbol{a}$, obtained by joining all the vertices of regular spherical polygon by the straight lines

In right $\Delta A_1MO'$

$$\Rightarrow \sin\angle A_1O'A_2 = \frac{A_1M}{O'A_1} \quad \Rightarrow \quad sin\frac{2\pi}{n} = \frac{A_1M}{\frac{Rsin\frac{\alpha}{2}}{sin\frac{\pi}{n}}}$$

$$\Rightarrow \boldsymbol{A_1M} = \frac{Rsin\frac{2\pi}{n}sin\frac{\alpha}{2}}{sin\frac{\pi}{n}} = \frac{2Rsin\frac{\pi}{n}cos\frac{\pi}{n}sin\frac{\alpha}{2}}{sin\frac{\pi}{n}}$$

$$= \boldsymbol{2Rsin\frac{\alpha}{2}cos\frac{\pi}{n}}$$

Now, join the points A_1 & M to the centre O of the sphere to obtain a right ΔA_1MO (see figure 4 below)

In right ΔA_1MO

$$\sin\angle A_1O'A_2 = \frac{A_1M}{OA_1}$$

$$\Rightarrow \quad sin\beta = \frac{2Rsin\frac{\alpha}{2}cos\frac{\pi}{n}}{R}$$

$$\boldsymbol{sin\beta = 2sin\frac{\alpha}{2}cos\frac{\pi}{n}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(IV)$$

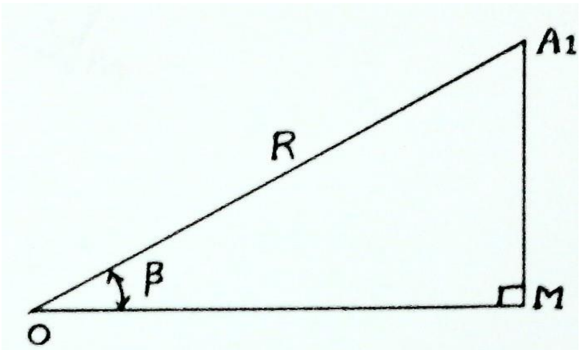

Figure 4: A right ΔA_1MO obtained by joining the points A_1 & M to the centre of the sphere with radius R

Now, comparing the equations (II) & (IV), we get

$$sin\beta = sin\alpha sin\frac{\theta}{2} = 2sin\frac{\alpha}{2}cos\frac{\pi}{n}$$

$$\Rightarrow 2sin\frac{\alpha}{2}cos\frac{\alpha}{2}sin\frac{\theta}{2} = 2sin\frac{\alpha}{2}cos\frac{\pi}{n}$$

$$cos\frac{\alpha}{2}sin\frac{\theta}{2} = cos\frac{\pi}{n}$$

$$\Rightarrow \ cos\frac{\frac{a}{R}}{2}sin\frac{\theta}{2} = cos\frac{\pi}{n} \qquad (on\ setting\ the\ value\ of\ \alpha\ from\ eq(I))$$

$$\boldsymbol{cos\frac{a}{2R}sin\frac{\theta}{2} = cos\frac{\pi}{n}} \qquad \ldots\ldots\ldots\ldots\ldots(V)$$

$$or\ \frac{a}{2R} = cos^{-1}\left(\frac{cos\frac{\pi}{n}}{sin\frac{\theta}{2}}\right)$$

$$\Rightarrow \boldsymbol{a = 2Rcos^{-1}\left(cos\frac{\pi}{n}cosec\frac{\theta}{2}\right)} \quad \forall\ \boldsymbol{\frac{(n-2)\pi}{n} < \theta < \pi\ \ \&\ n \geq 3} \quad \dots\dots\dots\dots\dots\dots..(VI)$$

Let's call the above relation (V) or (VI) as **HCR's Equation** i.e. **Characteristic Equation of Regular Spherical Polygon** which is extremely useful for calculating any of four important parameters $R, n, \theta\ \&\ a$ of any regular spherical polygon when rest three are known. Although the no. of sides n of regular spherical polygon is decided first since it is always a positive integer greter than 2 which can never be a fraction in any calculation. Thus three parameters (always considering n) are decided/optimized first arbitrarily and fourth one is calculated by above mathematical relation (from eq(V) or (VI)). It is extremely useful for drawing any regular n-polygon on the surface of a sphere having certain radius.

3. Application of HCR's Theory of Polygon to calculate solid angle (ω) subtended by a regular spherical polygon at the centre of the sphere:

We know from **HCR's theory of polygon, solid angle** (ω), **subtended by a regular polygon (i.e. plane bounded by straight lines) having n no. of sides each of length a at any point lying at a normal distance (height) H from the centre of plane,** is given as

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\frac{2Hsin\frac{\pi}{n}}{\sqrt{4H^2 + a^2cot^2\frac{\pi}{n}}}\right)}$$

If α is the angle between the consecutive lateral edges of the right pyramid obtained by joining all the vertices of regular polygon to the given point lying on the vertical axis passing through the centre of polygon then normal height (H) of right pyramid is given as

$$H = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}$$

$$\therefore\ \omega = 2\pi - 2n\sin^{-1}\left(\frac{2\left(\frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}\right)sin\frac{\pi}{n}}{\sqrt{4\left(\frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}\right)^2 + a^2cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{sin\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}}{\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n} + cot^2\frac{\pi}{n}}}\right) = 2\pi - 2n\sin^{-1}\left(\frac{sin\frac{\pi}{n}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}}{cot\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{sin\frac{\pi}{n}\sqrt{\frac{1}{tan^2\frac{\alpha}{2}} - \frac{1}{tan^2\frac{\pi}{n}}}}{cot\frac{\alpha}{2}}\right) = 2\pi - 2n\sin^{-1}\left(\frac{sin\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}}{cot\frac{\alpha}{2}tan\frac{\alpha}{2}tan\frac{\pi}{n}}\right)$$

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right)} \quad \dots\dots\dots\dots\dots\dots\dots\dots(VII)$$

Now, the value of **parametric angle α** can be substituted in the eq(VII) by two cases, depending on which parameter is known i.e. either length of each side (a) or magnitude of each interior angle (θ) while the radius of sphere (R) & the no. of sides (n) are already known/decided, as follows

Case 1: When the length a of side of regular spherical polygon is known: Then the value of **parametric angle α** is given from eq(I) as follows

$$\boldsymbol{\alpha = \frac{a}{R}}$$

a. Solid angle subtended by the regular spherical polygon at the centre of sphere:
By substituting the value of α in eq(VII), we get the value of solid angle as follows

$$\Rightarrow \ \omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right) = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2R}}\right)$$

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2R}}\right)}$$

b. Area of the regular spherical polygon: It is calculated by multiplying solid angle to the square of radius of the sphere as follows

$$Area\ of\ regular\ spherical\ polygon, A = (solid\ angle) \times (radius\ of\ sphere)^2 = \omega R^2$$

$$\boldsymbol{A = R^2\left[2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2R}}\right)\right]}$$

c. Solid angle subtended by the corresponding regular plane polygon at the centre of sphere:
Solid angle subtended by the corresponding regular plane polygon, obtained by consecutively joining all the vertices of a regular spherical polygon by straight lines, at the centre of the sphere is always equal to the solid angle subtended by the regular spherical polygon which is given as

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2R}}\right)}$$

d. Length of each side of the corresponding regular plane polygon: The **length of each side (a') of the corresponding regular plane polygon** is given from eq(III) as follows

$$a' = 2R\sin\frac{\alpha}{2} = 2R\sin\frac{\left(\frac{a}{R}\right)}{2} \quad \Rightarrow \quad \boldsymbol{a' = 2R\sin\frac{a}{2R}}$$

e. Normal height of the corresponding regular plane polygon from the centre of sphere:
Normal height $(\boldsymbol{H})$ of the corresponding regular plane polygon is calculated as follows

$$\boldsymbol{H} = \sqrt{R^2 - (circumscribed\ radius\ of\ regular\ plane\ polygon)^2}$$

$$= \sqrt{R^2 - \left(\frac{a'}{2\sin\frac{\pi}{n}}\right)^2} = \sqrt{R^2 - \left(\frac{2R\sin\frac{a}{2R}}{2\sin\frac{\pi}{n}}\right)^2} = R\sqrt{1 - \sin^2\frac{a}{2R}cosec^2\frac{\pi}{n}}$$

$$\boldsymbol{\therefore \ \ H = R\sqrt{1 - \sin^2\frac{a}{2R}cosec^2\frac{\pi}{n}}}$$

f. **Area of the corresponding regular plane polygon:** The **area (A') of the corresponding regular plane polygon** is given by the formula as

$$\boldsymbol{A'} = \frac{1}{4}n(a')^2 cot\frac{\pi}{n} \quad (generalized\ formula\ for\ any\ regular\ plane\ polygon)$$

$$= \frac{1}{4}n\left(2Rsin\frac{a}{2R}\right)^2 cot\frac{\pi}{n} = nR^2sin^2\frac{a}{2R}cot\frac{\pi}{n}$$

$$\boldsymbol{A' = nR^2sin^2\frac{a}{2R}cot\frac{\pi}{n}}$$

Case 2: When the interior angle θ of regular spherical polygon is known: The value of α is given from eq(V) as follows

$$cos\frac{a}{2R}sin\frac{\theta}{2} = cos\frac{\pi}{n} \Rightarrow cos\frac{\alpha}{2}sin\frac{\theta}{2} = cos\frac{\pi}{n} \qquad (since,\ \ \alpha = {}^{a}/_{R})$$

$$\Rightarrow \boldsymbol{cos\frac{\alpha}{2} = \frac{cos\frac{\pi}{n}}{sin\frac{\theta}{2}}}$$

$$\therefore\ tan^2\frac{\alpha}{2} = sec^2\frac{\alpha}{2} - 1 = \left(\frac{1}{cos\frac{\alpha}{2}}\right)^2 - 1 = \left(\frac{sin\frac{\theta}{2}}{cos\frac{\pi}{n}}\right)^2 - 1 = \frac{sin^2\frac{\theta}{2} - cos^2\frac{\pi}{n}}{cos^2\frac{\pi}{n}}$$

a. **Solid angle subtended by the regular spherical polygon at the centre of sphere:** By substituting the value of $tan^2\,{}^{\alpha}/_{2}$ in the eq(VII), we get the solid angle as follows

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right) = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - \frac{sin^2\frac{\theta}{2} - cos^2\frac{\pi}{n}}{cos^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{sin^2\frac{\pi}{n} - sin^2\frac{\theta}{2} + cos^2\frac{\pi}{n}}\right) = 2\pi - 2n\sin^{-1}\left(\sqrt{1 - sin^2\frac{\theta}{2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{cos^2\frac{\theta}{2}}\right) = 2\pi - 2n\sin^{-1}\left(cos\frac{\theta}{2}\right) = 2\pi - 2n\sin^{-1}\left(sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) \qquad (since,\ \ \theta < \pi)$$

$$= 2\pi - 2n\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 2\pi - n(\pi - \theta)$$

$$\therefore\ \boldsymbol{\omega = 2\pi - n(\pi - \theta)}$$

b. **Area of the regular spherical polygon:** It is calculated by multiplying solid angle to the square of radius of the sphere as follows

$$Area\ of\ regular\ spherical\ polygon, A = \omega R^2$$

$$\boldsymbol{A = R^2[2\pi - n(\pi - \theta)]}$$

c. Solid angle subtended by the corresponding regular plane polygon at the centre of sphere: Solid angle subtended by the corresponding regular plane polygon at the centre of the sphere is always equal to the solid angle subtended by the regular spherical polygon which is given as

$$\boldsymbol{\omega = 2\pi - n(\pi - \theta)}$$

d. Length of each side of the corresponding regular plane polygon: The **length of each side** $(\boldsymbol{a'})$ **of the regular plane polygon** is given from eq(III) as follows

$$a' = 2R\sin\frac{\alpha}{2} = 2R\sqrt{1-\cos^2\frac{\alpha}{2}} = 2R\sqrt{1-\left(\frac{\cos\frac{\pi}{n}}{\sin\frac{\theta}{2}}\right)^2} = 2R\sqrt{1-\cos^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}}$$

$$\Rightarrow \boldsymbol{a' = 2R\sqrt{1-\cos^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}}}$$

e. Normal height of the corresponding regular plane polygon from the centre of sphere: Normal height of the corresponding regular plane polygon from the centre of sphere is calculated as follows

$$\boldsymbol{H} = \sqrt{R^2 - (\text{circumscribed radius of regular plane polygon})^2}$$

$$= \sqrt{R^2 - \left(\frac{a'}{2\sin\frac{\pi}{n}}\right)^2} = \sqrt{R^2 - \left(\frac{2R\sqrt{1-\cos^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}}}{2\sin\frac{\pi}{n}}\right)^2} = R\sqrt{1-\left(\frac{1-\cos^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}}{\sin^2\frac{\pi}{n}}\right)}$$

$$= R\sqrt{1-\operatorname{cosec}^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}} = R\sqrt{\cot^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}-\cot^2\frac{\pi}{n}} = R\cot\frac{\pi}{n}\sqrt{\operatorname{cosec}^2\frac{\theta}{2}-1}$$

$$= R\cot\frac{\pi}{n}\sqrt{\cot^2\frac{\theta}{2}} = R\cot\frac{\pi}{n}\cot\frac{\theta}{2}$$

$$\therefore\ \boldsymbol{H = R\cot\frac{\pi}{n}\cot\frac{\theta}{2}}$$

f. Area of the corresponding regular plane polygon: The **area** $(\boldsymbol{A'})$ **of the corresponding regular plane polygon** is given by the formula as

$$\boldsymbol{A'} = \frac{1}{4}n(a')^2\cot\frac{\pi}{n} \qquad (\text{generalized formula for any regular plane polygon})$$

$$= \frac{1}{4}n\left(2R\sqrt{1-\cos^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}}\right)^2\cot\frac{\pi}{n} = nR^2\left(1-\cos^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}\right)\cot\frac{\pi}{n}$$

$$\boldsymbol{A' = nR^2\left(1-\cos^2\frac{\pi}{n}\operatorname{cosec}^2\frac{\theta}{2}\right)\cot\frac{\pi}{n} \quad \forall\ \frac{(n-2)\pi}{n} < \theta < \pi\ \&\ n \geq 3}$$

4. Regular spherical polygon on the unit sphere (radius, $R = 1$)

Let there be any regular polygon having n no. of the sides each of length a & each interior angle θ. Then all the important parameters of the regular spherical polygon on the unit sphere can be obtained very simply by substituting $R = 1$ in all the expressions of above two cases as follows

Case 1: When the length a of side of regular spherical polygon is known: Then the value parametric angle of α is given from eq(I) as follows

$$\alpha = \frac{a}{R} = \frac{a}{1} \quad \Rightarrow \boldsymbol{\alpha = a} \quad (\boldsymbol{in\ radian})$$

a. Solid angle subtended by the regular spherical polygon at the centre of unit sphere: By substituting the value of α in eq(VII), we get the value of solid angle as follows

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right) = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2}}\right)$$

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2}}\right)}$$

b. Area of the regular spherical polygon: It is calculated by multiplying solid angle to the square of radius of the unit sphere as follows

$$Area\ of\ regular\ spherical\ polygon,\ \ A = (solid\ angle) \times (radius\ of\ unit\ sphere)^2 = \omega(1)^2 = \omega$$

$$\boldsymbol{A = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2}}\right)}$$

c. Solid angle subtended by the corresponding regular plane polygon at the centre of unit sphere: Solid angle subtended by the regular plane polygon, obtained by consecutively joining all the vertices of a regular spherical polygon by straight lines, at the centre of unit sphere is always equal to the solid angle subtended by the regular spherical polygon which is given as follows

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{a}{2}}\right)}$$

d. Length of each side of the corresponding regular plane polygon: The **length of side** (a') **of the corresponding regular plane polygon** is given from eq(III) as follows

$$a' = 2R\sin\frac{\alpha}{2} \quad \Rightarrow a' = 2(1)\sin\frac{a}{2} = 2\sin\frac{a}{2} \quad \Rightarrow \boldsymbol{a' = 2\sin\frac{a}{2}}$$

e. Normal height of the corresponding regular plane polygon from the centre of unit sphere:

Normal height $(\boldsymbol{H})$ of the corresponding regular plane polygon from the centre of unit sphere is calculated as follows

$$\boldsymbol{H} = \sqrt{R^2 - (circumscribed\ radius\ of\ regular\ plane\ polygon)^2}$$

$$= \sqrt{R^2 - \left(\frac{a'}{2\sin\frac{\pi}{n}}\right)^2} = \sqrt{(1)^2 - \left(\frac{2\sin\frac{a}{2}}{2\sin\frac{\pi}{n}}\right)^2} = \sqrt{1 - \sin^2\frac{a}{2}\mathrm{cosec}^2\frac{\pi}{n}}$$

$$\therefore\ \boldsymbol{H = \sqrt{1 - \sin^2\frac{a}{2}\mathrm{cosec}^2\frac{\pi}{n}}}$$

f. **Area of the corresponding regular plane polygon:** The **area (A') of the corresponding regular plane polygon** is given by the formula as

$$\boldsymbol{A'} = \frac{1}{4}n(a')^2 cot\frac{\pi}{n} \quad (generalized\ formula\ for\ any\ regular\ plane\ polygon)$$

$$= \frac{1}{4}n\left(2sin\frac{a}{2}\right)^2 cot\frac{\pi}{n} = nsin^2\frac{a}{2}cot\frac{\pi}{n}$$

$$\boldsymbol{A' = nsin^2\frac{a}{2}cot\frac{\pi}{n}}$$

Case 2: When the interior angle θ of regular spherical polygon is known: The value of α is given from eq(V) as follows

$$cos\frac{a}{2R}sin\frac{\theta}{2} = cos\frac{\pi}{n} \Rightarrow cos\frac{a}{2}sin\frac{\theta}{2} = cos\frac{\pi}{n} \qquad (since, R = 1)$$

$$\Rightarrow \boldsymbol{cos\frac{\alpha}{2} = \frac{cos\frac{\pi}{n}}{sin\frac{\theta}{2}}} \qquad \left(since,\ \alpha = {}^{a}/_{R} = {}^{a}/_{1} = a\right)$$

$$\therefore\ tan^2\frac{\alpha}{2} = sec^2\frac{\alpha}{2} - 1 = \left(\frac{1}{cos\frac{\alpha}{2}}\right)^2 - 1 = \left(\frac{sin\frac{\theta}{2}}{cos\frac{\pi}{n}}\right)^2 - 1 = \frac{sin^2\frac{\theta}{2} - cos^2\frac{\pi}{n}}{cos^2\frac{\pi}{n}}$$

a. **Solid angle subtended by the regular spherical polygon at the centre of unit sphere:** By substituting the value of $tan^2\,{}^{\alpha}/_{2}$ in the eq(VII), we get the solid angle as follows

$$\Rightarrow \omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}\right) = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - \frac{sin^2\frac{\theta}{2} - cos^2\frac{\pi}{n}}{cos^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{sin^2\frac{\pi}{n} - sin^2\frac{\theta}{2} + cos^2\frac{\pi}{n}}\right) = 2\pi - 2n\sin^{-1}\left(\sqrt{1 - sin^2\frac{\theta}{2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(cos\frac{\theta}{2}\right) = 2\pi - 2n\sin^{-1}\left(sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right) = 2\pi - 2n\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = 2\pi - n(\pi - \theta)$$

$$\therefore\ \boldsymbol{\omega = 2\pi - n(\pi - \theta)}$$

b. **Area of the regular spherical polygon:** It is calculated by multiplying solid angle to the square of radius of the unit sphere as follows

$$Area\ of\ regular\ spherical\ polygon, A = \omega R^2 = \omega(1)^2 = \omega$$

$$\boldsymbol{A = 2\pi - n(\pi - \theta)}$$

c. **Solid angle subtended by the corresponding regular plane polygon at the centre of unit sphere:** Solid angle subtended by the corresponding regular plane polygon, obtained by consecutively joining all the vertices of a regular spherical polygon by straight lines, at the centre of unit sphere is always equal to the solid angle subtended by the regular spherical polygon which is given as

$$\boldsymbol{\omega = 2\pi - n(\pi - \theta)}$$

d. **Length of each side of the corresponding regular plane polygon:** The **length of each side** $(\boldsymbol{a'})$ **of the corresponding regular plane polygon,** obtained by consecutively joining all the vertices of a regular spherical polygon by straight lines, is given from eq(III) as follows

$$a' = 2R\sin\frac{\alpha}{2} = 2\sqrt{1-\cos^2\frac{\alpha}{2}} = 2\sqrt{1-\left(\frac{\cos\frac{\pi}{n}}{\sin\frac{\theta}{2}}\right)^2} = 2\sqrt{1-\cos^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}}$$

$$\Rightarrow \boldsymbol{a' = 2\sqrt{1-\cos^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}}}$$

e. **Normal height of the corresponding regular plane polygon from the centre of unit sphere:**

Normal height $(\boldsymbol{H})$ of the corresponding regular plane polygon from the centre of unit sphere is calculated as follows

$$\boldsymbol{H} = \sqrt{R^2 - (circumscribed\ radius\ of\ regular\ plane\ polygon)^2}$$

$$= \sqrt{R^2 - \left(\frac{a'}{2\sin\frac{\pi}{n}}\right)^2} = \sqrt{(1)^2 - \left(\frac{2\sqrt{1-\cos^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}}}{2\sin\frac{\pi}{n}}\right)^2} = \sqrt{1-\left(\frac{1-\cos^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}}{\sin^2\frac{\pi}{n}}\right)}$$

$$= \sqrt{1-\mathrm{cosec}^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}} = \sqrt{\cot^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}-\cot^2\frac{\pi}{n}} = \cot\frac{\pi}{n}\sqrt{\mathrm{cosec}^2\frac{\theta}{2}-1}$$

$$= \cot\frac{\pi}{n}\sqrt{\cot^2\frac{\theta}{2}} = \cot\frac{\pi}{n}\cot\frac{\theta}{2}$$

$$\therefore\ \boldsymbol{H = \cot\frac{\pi}{n}\cot\frac{\theta}{2}}$$

f. **Area of the corresponding regular plane polygon:** The **area** $(\boldsymbol{A'})$ **of the corresponding regular plane polygon** is given by the formula as

$$\boldsymbol{A'} = \frac{1}{4}n(a')^2\cot\frac{\pi}{n} \qquad (generalized\ formula\ for\ any\ regular\ plane\ polygon)$$

$$= \frac{1}{4}n\left(2\sqrt{1-\cos^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}}\right)^2\cot\frac{\pi}{n} = n\left(1-\cos^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}\right)\cot\frac{\pi}{n}$$

$$\boldsymbol{A' = n\left(1-\cos^2\frac{\pi}{n}\mathrm{cosec}^2\frac{\theta}{2}\right)\cot\frac{\pi}{n} \quad \forall\ \frac{(n-2)\pi}{n} < \theta < \pi\ \&\ n \geq 3}$$

5. Solution of the Greatest Regular Spherical Polygon (a regular spherical polygon having maximum no. of sides) for a given value of interior angle $(\boldsymbol{\theta < \pi})$

Suppose we are/have to draw a regular spherical polygon with the maximum no. of sides (n) such that each of its interior angle is $\theta(<\pi)$ then the maximum no. of sides (n_{max}) can be easily calculated by the following inequality as follows

$$\frac{(n-2)\pi}{n} < \theta < \pi \quad let's\ consider,\ \frac{(n-2)\pi}{n} < \theta$$

$$\Rightarrow \quad n\pi - 2\pi < n\theta \quad or \quad n(\pi - \theta) < 2\pi \qquad \Rightarrow n < \frac{2\pi}{\pi - \theta}$$

Hence the maximum no. of the sides (n_{max}) of the greatest regular is calculated by the following inequality

$$\therefore \ \boldsymbol{n_{max} < \frac{2\pi}{\pi - \theta} \quad \forall \ \theta < \pi}$$

$$\boldsymbol{or \quad n_{max} < \frac{360^o}{180^o - \theta} \quad \forall \ \theta < 180^o}$$

After calculating the maximum of no. of sides (n_{max}) of the regular spherical polygon with known interior angle (θ) there arise two cases as follows

Case 1: When the radius (R) of the sphere is known/given: In this case all the parameters of the **greatest regular spherical polygon** are calculated as follows

Minimum length of side(a_{min}): The minimum length of side (a_{min}) of the greatest regular spherical polygon is simply calculated by using characteristic equation (given from eq(VI)) as follows

$$a = 2Rcos^{-1}\left(cos\frac{\pi}{n}cosec\frac{\theta}{2}\right) \quad \Rightarrow \ \boldsymbol{a_{min} = 2Rcos^{-1}\left(cos\frac{\pi}{n_{max}}cosec\frac{\theta}{2}\right)}$$

Minimum solid angle(ω_{min}): We know that solid angle subtended by a regular spherical n-polygon with each interior angle θ is given by the relation of case -2 as follows

$$\omega = 2\pi - n(\pi - \theta) \quad \Rightarrow \ \boldsymbol{\omega_{min} = 2\pi - n_{max}(\pi - \theta)}$$

Minimum area (A_{min}): The area of the greatest regular spherical polygon with n_{max} no. of sides & each interior angle θ drawn on the sphere of radius R, is given as follows

$$A = \omega R^2 \quad \Rightarrow \ \boldsymbol{A_{min} = R^2\left(2\pi - n_{max}(\pi - \theta)\right)}$$

Case 2: When the length of side (a) of the greatest regular spherical polygon is known/given: In this case all the parameters of the **greatest regular spherical polygon** are calculated as follows

Radius of the sphere (R): The radius (R) of the sphere, on which the greatest regular spherical polygon can be drawn/traced, is simply calculated by using **characteristic equation** (given from eq(VI)) as follows

$$a = 2Rcos^{-1}\left(cos\frac{\pi}{n}cosec\frac{\theta}{2}\right) \quad \Rightarrow \ \boldsymbol{R = \frac{a}{2cos^{-1}\left(cos\frac{\pi}{n_{max}}cosec\frac{\theta}{2}\right)}}$$

Note: above value of the radius R is maximum for known values of θ, n_{max} & a but it's the **required value of the radius of the sphere on which a regular polygon with n_{max} no. of sides each of length a & each interior angle θ can be drawn/traced**. It is because any regular polygon with known values of three parameters θ, n_{max} & a can never be drawn on the surface of sphere having a **radius theoretically different from above required value of R** i.e. for known values of three parameters of out of four parameters of any regular spherical polygon, the forth one is always unique which is always calculated from the **characteristic equation of regular spherical polygon.**

Minimum solid angle(ω_{min}): We know that solid angle subtended by a regular spherical n-polygon with each interior angle θ is given by the relation of case -2 as follows

$$\omega = 2\pi - n(\pi - \theta) \ \Rightarrow \ \boldsymbol{\omega_{min} = 2\pi - n_{max}(\pi - \theta)}$$

Minimum area (A_{min}): The area of the greatest regular spherical polygon with n_{max} no. of sides & each interior angle θ drawn on the sphere of radius R, is given as follows

$$A = \omega R^2 \quad \Rightarrow \quad A_{min} = \left(\frac{a}{2\cos^{-1}\left(\cos\frac{\pi}{n_{max}}\operatorname{cosec}\frac{\theta}{2}\right)}\right)^2 \left(2\pi - n_{max}(\pi - \theta)\right)$$

$$= \left(\frac{a}{2\cos^{-1}\left(\cos\frac{\pi}{n_{max}}\operatorname{cosec}\frac{\theta}{2}\right)}\right)^2 = \frac{a^2\left(2\pi - n_{max}(\pi - \theta)\right)}{4\left(\cos^{-1}\left(\cos\frac{\pi}{n_{max}}\operatorname{cosec}\frac{\theta}{2}\right)\right)^2} = \frac{a^2}{4}\left(\frac{2\pi - n_{max}(\pi - \theta)}{\left(\cos^{-1}\left(\cos\frac{\pi}{n_{max}}\operatorname{cosec}\frac{\theta}{2}\right)\right)^2}\right)$$

$$\Rightarrow \boldsymbol{A_{min} = \frac{a^2}{4}\left(\frac{2\pi - n_{max}(\pi - \theta)}{\left(\cos^{-1}\left(\cos\frac{\pi}{n_{max}}\operatorname{cosec}\frac{\theta}{2}\right)\right)^2}\right)}$$

6. Working steps to draw any regular n-polygon on the surface of a sphere having certain radius:

We know that any three parameters (always considering n i.e. no. of sides of polygon) out of four important parameters R, n, θ & a of any regular spherical polygon are **decided/optimized arbitrarily** as required and then fourth one is calculated by the mathematical relation (characteristic equation) which is given from eq(V) or (VI). **Let there be a sphere with certain radius R (known) then let's follow the steps below to draw any regular polygon on the surface of this sphere**

Step 1: First of all decide the **no. of sides $\boldsymbol{n}$** of the regular polygon $\forall\, \boldsymbol{n \geq 3}$ to be drawn on the spherical surface. It's because n is always a positive integer which must not be a fraction in any calculation hence we are constrained to decide it first.

Step 2: Then decide/optimize the **value of interior angle $\boldsymbol{\theta}$** of the spherical regular polygon for decided or given value of n (i.e. no. of sides of polygon) as follows

$$\boldsymbol{\frac{(n-2)\pi}{n} < \theta < \pi}$$

Step 3: Calculate the **length of each side $(\boldsymbol{a})$ of the regular spherical polygon** using the relation from eq(VI) as follows

$$\boldsymbol{a = 2R\cos^{-1}\left(\cos\frac{\pi}{n}\operatorname{cosec}\frac{\theta}{2}\right)}$$

Step 4: Now draw/trace n no. of sides each of length a & each interior angle θ to obtain regular spherical polygon with known parameters.

Note: In order to calculate other parameters such as solid angle & area, use the necessary equations above.

7. Working steps to draw the greatest regular spherical polygon, for a given value of its interior angle $(\boldsymbol{\theta < \pi \text{ or } 180^o})$, on the surface of a sphere with a known/given radius: Let there be sphere with a known radius R on which we are to draw a regular polygon having the maximum no. of sides (n_{max}) such that each of its interior angles is θ then we should follows the steps below

Step 1: In order to draw/trace the greatest regular spherical polygon (having maximum no. of sides) for the given value of interior angle θ then first calculate maximum no. of sides $n_{max}$by using inequality as follows

$$\therefore\ \boldsymbol{n_{max} < \frac{2\pi}{\pi-\theta} \quad \forall\ \theta < \pi}$$

$$\boldsymbol{or \quad n_{max} < \frac{360^o}{180^o-\theta} \quad \forall\ \theta < 180^o}$$

Step 2: Calculate the **length of each side** $(\boldsymbol{a})$ **of the greatest regular spherical polygon** using the relation from as follows

$$\boldsymbol{a_{min} = 2Rcos^{-1}\left(cos\frac{\pi}{n_{max}}cosec\frac{\theta}{2}\right)}$$

Step 4: Now draw/trace n_{max} no. of sides each of length a_{min} & each interior angle θ to obtain **the greatest regular spherical polygon** with known parameters.

Note: In order to calculate other parameters such as solid angle & area, use the necessary equations above.

Illustrative Numerical Examples

These examples are based on all above articles which very practical & directly & simply applicable to calculate the different parameters of any regular spherical polygon.

Example 1: **Calculate the area & each of the interior angles of a regular spherical polygon, having 15 no. of sides each of length 12 units, drawn on the sphere with a radius of 100 units.**

Sol. Here, we have

$$R = 100\ units, \ \ n = 15 \ \ \& \ \ a = 12\ units \ \Rightarrow \ A = ?\ \& \ \ \theta = ?$$

We know the relation of four parameters of a regular spherical polygon from characteristic equation as

$$cos\frac{a}{2R}sin\frac{\theta}{2} = cos\frac{\pi}{n} \qquad\qquad (from\ eq(V))$$

$$\Rightarrow cos\frac{12}{200}sin\frac{\theta}{2} = cos\frac{\pi}{15} \ \Rightarrow \ \ sin\frac{\theta}{2} = \frac{cos\frac{\pi}{15}}{cos\frac{3}{50}} \qquad or \ \ \theta = 2sin^{-1}\left(\frac{cos\frac{\pi}{15}}{cos\frac{3}{50}}\right)$$

$$\therefore \ \ \boldsymbol{\theta} = 2sin^{-1}\left(\frac{cos\frac{\pi}{15}}{cos\frac{3}{50}}\right) \approx \boldsymbol{156.9920733^o \approx 156^o59'31.46''}$$

Hence, the area of regular spherical polygon $(\boldsymbol{n = 15})$ is given as follows

$$A = R^2\left[2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{a}{2R}}\right)\right]$$

$$= (100)^2\left[2\pi - 2(15)\sin^{-1}\left(cos\frac{\pi}{15}\sqrt{tan^2\frac{\pi}{15} - tan^2\frac{12}{200}}\right)\right]$$

$$= 10000\left[2\pi - 30\sin^{-1}\left(cos\frac{\pi}{15}\sqrt{tan^2\frac{\pi}{15} - tan^2\frac{3}{50}}\right)\right] \approx \mathbf{2597.241808\, unit^2}$$

$$\therefore\ solid\ angle\ subtended\ by\ the\ regular\ spherical\ polygon, \omega = \frac{A}{R^2} = \frac{2597.241808}{(100)^2}$$

$$\approx 0.2597241808\ sr$$

The above value of area (A) implies that the given regular polygon $(n = 15)$ covers approximately $\mathbf{2597.241808\, unit^2}$ of the total surface area $= \mathbf{4\pi(100)^2} \approx \mathbf{125663.7061\, unit^2}$ of the sphere with a radius of 100 units i.e. the **regular polygon $(\boldsymbol{n = 15})$ covers approximately $\mathbf{0.020668193}$ times the total surface area & subtends a solid angle $\mathbf{0.2597241808\, sr}$ at the centre of the sphere with a radius of 100 units.**

Example 2: **Calculate area & length of each side of a regular spherical hexagon, having each interior angle $\mathbf{150^o}$, drawn on the sphere with a radius of 60 units.**

Sol. Here, we have

$$R = 60\, units,\ \ n = 6\ (for\ regular\ hexagon)\ \&\ \theta = 150^o = \frac{5\pi}{6} \Rightarrow A = ?\ \&\ a = ?$$

$$\therefore\ \boldsymbol{each\ side\ of\ regular\ spherical\ hexagon},\ \ a = 2Rcos^{-1}\left(cos\frac{\pi}{n}cosec\frac{\theta}{2}\right)$$

$$= 2(60)cos^{-1}\left(cos\frac{\pi}{6}cosec\frac{5\pi}{12}\right) = 120cos^{-1}\left(cos\frac{\pi}{6}cosec\frac{5\pi}{12}\right) \approx \mathbf{55.05847049\, units}$$

$$\therefore \boldsymbol{Area\ of\ regular\ spherical\ hexagon},\ \ A = R^2[2\pi - n(\pi - \theta)]$$

$$= (60)^2\left[2\pi - 6\left(\pi - \frac{5\pi}{6}\right)\right] = 3600\left[2\pi - 6\left(\frac{\pi}{6}\right)\right] = 3600[2\pi - \pi] = 3600\pi \approx \mathbf{11309.73355\, unit^2}$$

$$\therefore\ solid\ angle\ subtended\ by\ the\ regular\ spherical\ hexagon, \omega = \frac{A}{R^2} = \frac{3600\pi}{(60)^2} = \pi\ sr$$

The above value of area (A) implies that the given regular hexagon covers $\boldsymbol{\pi(60)^2} \approx \mathbf{11309.73355\, unit^2}$ of the total surface area $= \boldsymbol{4\pi(60)^2} \approx \mathbf{45238.93421\, unit^2}$ of the sphere with a radius of 60 units i.e. the **regular hexagon covers exactly $\mathbf{1/4}$ of the total surface area & subtends a solid angle $\boldsymbol{\pi\, sr}$ at the centre of the sphere with a radius of 60 units.**

Example 3: **Calculate area & length of each side of a regular spherical polygon having maximum no. of sides such that each of its interior angle is $\mathbf{170^o}$, drawn on the sphere with a radius of 200 units.**

Sol. Here, we have

$$R = 200\, units\ \ \&\ \theta = 170^o = \frac{17\pi}{18}\ \ \Rightarrow A = ?\ \&\ a = ?$$

In this case, maximum possible no. of sides (a_{max}) is calculated by the following inequality

$$n_{max} < \frac{2\pi}{\pi - \theta}\ \Rightarrow n_{max} < \frac{2\pi}{\pi - \frac{17\pi}{18}}\ \ or\ \ n_{max} < \frac{36\pi}{\pi}\ \ or\ n_{max} < 36$$

$$\therefore\ \boldsymbol{n_{max} = 35}$$

$$\therefore\ \boldsymbol{each\ side\ of\ regular\ spherical\ polygon}\ (\boldsymbol{n_{max} = 35}),\ \ a = 2Rcos^{-1}\left(cos\frac{\pi}{n_{max}}cosec\frac{\theta}{2}\right)$$

$$= 2(200)cos^{-1}\left(cos\frac{\pi}{35}cosec\frac{17\pi}{36}\right) = 400cos^{-1}\left(cos\frac{\pi}{35}cosec\frac{17\pi}{36}\right) \approx \boldsymbol{8.414352743\ units}$$

$$\therefore \boldsymbol{Area\ of\ regular\ spherical\ polygon}\ \ (\boldsymbol{n_{max} = 35}),\ \ A = R^2[2\pi - n_{max}(\pi - \theta)]$$

$$= (200)^2\left[2\pi - 35\left(\pi - \frac{17\pi}{18}\right)\right] = 40000\left[2\pi - 35\left(\frac{\pi}{18}\right)\right] = 40000\left[\frac{36\pi - 35\pi}{18}\right]$$

$$= 40000\left(\frac{\pi}{18}\right) = \frac{20000\pi}{9} \approx \boldsymbol{6981.317008\ unit^2}$$

$$\therefore\ solid\ angle\ subtended\ by\ the\ regular\ spherical\ polygon, \omega = \frac{A}{R^2} = \frac{20000\pi}{9(200)^2} = \frac{\pi}{18}\ sr$$

The above value of area (A) implies that the given regular polygon ($n_{max} = 35$) covers $\boldsymbol{\pi(200)^2/_{18}} \approx \boldsymbol{6981.317008\ unit^2}$ of the total surface area $= \boldsymbol{4\pi(200)^2} \approx \boldsymbol{502654.8246\ unit^2}$ of the sphere with a radius of 200 units i.e. the **regular polygon ($\boldsymbol{n_{max} = 35}$) covers exactly $\boldsymbol{1/_{72}}$ of the total surface area & subtends a solid angle $\boldsymbol{\pi/_{18}\ sr}$ at the centre of the sphere with a radius of 200 units.**

Conclusion: *All the articles above have been derived by **Mr H.C. Rajpoot** (without using the vector analysis or any other method) only by using **simple geometry & trigonometry**. All above articles (formulae) are very practical & simple to apply in case of **any regular spherical polygon** to calculate all its important parameters such as solid angle, covered surface area & arc length of each side etc. & also useful for calculating all the parameters of the **corresponding regular plane polygon** obtained by joining all the vertices of the regular spherical polygon by straight lines. These formulae can also be used to calculate all the parameters of the right pyramid obtained by joining all the vertices of a regular spherical polygon to the centre of sphere such as normal height, angle between the consecutive lateral edges, area of base etc. All these results are also the shortcuts for solving the various complex problems related to the regular spherical polygons.*

Let there be any **regular spherical polygon**, having $\boldsymbol{n}$ no. of sides each (as an arc) of length $\boldsymbol{a}$ & each interior angle $\boldsymbol{\theta}$, drawn on the surface of the sphere with a radius $\boldsymbol{R}$ then solid angle subtended by it at the centre of sphere & the area covered by it are calculated simply by using three known parameters out of four parameters $\boldsymbol{R}, \boldsymbol{n}, \boldsymbol{\theta}$ & $\boldsymbol{a}$ & fourth one is calculated by the parametric relation as **briefly tabulated** below

Three known parameters	*Solid angle subtended by the regular spherical polygon at the centre of sphere (in Ste-radian)*	*Area covered by the regular spherical polygon*	*Fourth unknown parameter*
R, n & a	$\omega = 2\pi - 2n\sin^{-1}\left(cos\frac{\pi}{n}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{a}{2R}}\right)$	$A = \omega R^2$	$\theta = 2sin^{-1}\left(cos\frac{\pi}{n}sec\frac{a}{2R}\right)$
R, n & θ	$\omega = 2\pi - n(\pi - \theta)$	$A = \omega R^2$	$a = 2Rcos^{-1}\left(cos\frac{\pi}{n}cosec\frac{\theta}{2}\right)$

Note: Above articles had been derived *& illustrated by **Mr H.C. Rajpoot** (**B Tech, Mechanical Engineering**)*

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