## *Mathematical Analysis of Three Externally Touching Circles*

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### 1. Introduction:

Consider three circles having centres A, B & C and radii $a, b$ & $c$ respectively, touching each other externally such that a small circle P is inscribed in the gap & touches them externally & a large circle Q circumscribes them & is touched by them internally. We are to calculate the radii of **inscribed circle P** (touching three circles with centres A, B & C externally) & **circumscribed circle Q** (touched by three circles with centres A, B & C internally) (See figure 1)

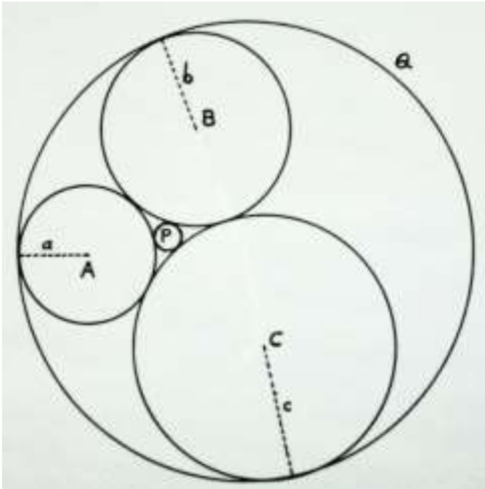


**Figure 1: Three circles with centres A, B & C and radii $a, b$ & $c$ respectively are touching each other externally. Inscribed circle P & circumscribed circle Q are touching these circles externally & internally**

**2. Derivation of the radius of inscribed circle:** Let $r$ be the radius of inscribed circle, with centre O, externally touching the given circles, having centres A, B & C and radii $a, b$ & $c$, at the points M, N & P respectively. Now join the centre O to the centres A, B & C by dotted straight lines to obtain $\Delta AOB$, $\Delta BOC$ & $\Delta AOC$ & also join the centres A, B & C by dotted straight lines to obtain $\Delta ABC$ (As shown in the figure 2 below) Thus we have

$$AM = a, \quad BN = b, \quad CP = c \;\; \&$$

$$OM = ON = OP = r \; (radius\ of\ inscribed\ circle)$$

In $\Delta ABC$

$$AB = a + b, \;\; BC = b + c \;\; \& \;\; AC = a + c$$

$$\Rightarrow semiperimeter = \frac{AB + BC + AC}{2}$$

$$s = \frac{a + b + b + c + a + c}{2} = a + b + c$$

$$sin\frac{\angle ACB}{2} = \sqrt{\frac{(s - AC)(s - BC)}{(AC)(BC)}}$$

$$\Rightarrow \; sin\frac{\alpha}{2} = \sqrt{\frac{(a + b + c - a - c)(a + b + c - b - c)}{(a + c)(b + c)}}$$

$$\boldsymbol{sin\frac{\alpha}{2} = \sqrt{\frac{ab}{(a + c)(b + c)}}} \quad \ldots\ldots\ldots.(\boldsymbol{I})$$

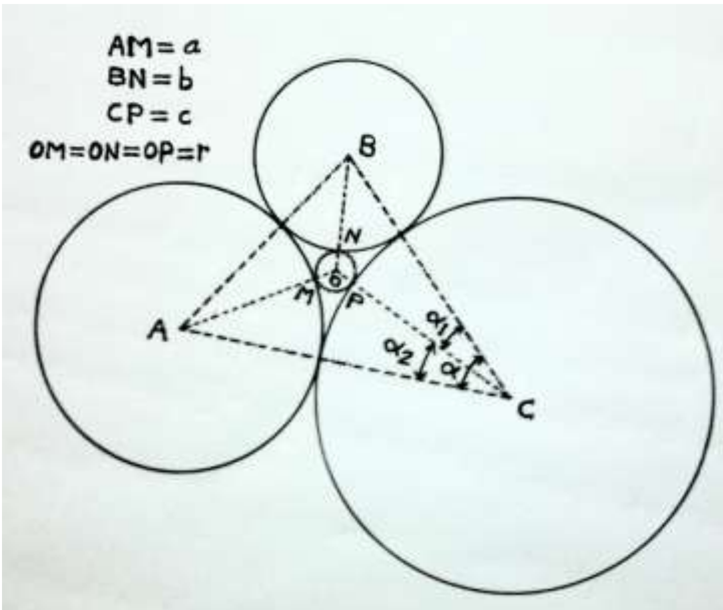


**Figure 2: The centres A, B, C & O are joined to each other by dotted straight lines to obtain $\Delta ABC, \Delta AOB, \Delta BOC$ & $\Delta AOC$**

Similarly, in $\Delta BOC$

$$semiperimeter = \frac{OB + BC + OC}{2} \Rightarrow s = \frac{r + b + b + c + r + c}{2} = b + c + r$$

$$\cos\frac{\angle BCO}{2}=\sqrt{\frac{s(s-OB)}{(BC)(OC)}}\ \Rightarrow \cos\frac{\alpha_1}{2}=\sqrt{\frac{(b+c+r)(b+c+r-r-b)}{(b+c)(r+c)}}=\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}}$$

$$\boldsymbol{\cos\frac{\alpha_1}{2}=\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}}}\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(\boldsymbol{II})$$

Similarly, in $\Delta AOC$

$$semiperimeter=\frac{OA+AC+OC}{2}\Rightarrow s=\frac{r+a+a+c+r+c}{2}=a+c+r$$

$$\cos\frac{\angle ACO}{2}=\sqrt{\frac{s(s-OA)}{(AC)(OC)}}\ \Rightarrow \cos\frac{\alpha_2}{2}=\sqrt{\frac{(a+c+r)(a+c+r-r-a)}{(a+c)(r+c)}}=\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}}$$

$$\boldsymbol{\cos\frac{\alpha_2}{2}=\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}}}\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(\boldsymbol{III})$$

Now, again in $\Delta ABC$, we have

$$\angle ACO+\angle BCO=\angle ACB\ \Rightarrow \alpha_2+\alpha_1=\alpha\quad or\quad \frac{\alpha_1}{2}+\frac{\alpha_2}{2}=\frac{\alpha}{2}$$

Now, taking **cosines** of both the sides we have

$$\cos\left(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}\right)=\cos\frac{\alpha}{2}\Rightarrow \cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}-\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}=\cos\frac{\alpha}{2}$$

$$\Rightarrow \left(\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}-\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\right)^2=\left(\cos\frac{\alpha}{2}\right)^2$$

$$\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}+\sin^2\frac{\alpha_1}{2}\sin^2\frac{\alpha_2}{2}-2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}=\cos^2\frac{\alpha}{2}$$

$$\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}+\left(1-\cos^2\frac{\alpha_1}{2}\right)\left(1-\cos^2\frac{\alpha_2}{2}\right)-\cos^2\frac{\alpha}{2}=2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}$$

$$\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}+1-\cos^2\frac{\alpha_1}{2}-\cos^2\frac{\alpha_1}{2}+\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-\cos^2\frac{\alpha}{2}=2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}$$

$$\Rightarrow 2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-\cos^2\frac{\alpha_1}{2}-\cos^2\frac{\alpha_1}{2}+\sin^2\frac{\alpha}{2}=2\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sqrt{\left(1-\cos^2\frac{\alpha_1}{2}\right)}\sqrt{\left(1-\cos^2\frac{\alpha_2}{2}\right)}$$

$$\Rightarrow \left(2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-\cos^2\frac{\alpha_1}{2}-\cos^2\frac{\alpha_1}{2}+\sin^2\frac{\alpha}{2}\right)^2=4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\left(1-\cos^2\frac{\alpha_1}{2}\right)\left(1-\cos^2\frac{\alpha_2}{2}\right)$$

$$\Rightarrow 4\cos^4\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}-2\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-2\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}+2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}-2\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}$$
$$+\cos^4\frac{\alpha_1}{2}+\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2}-2\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}+\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}$$
$$+\cos^4\frac{\alpha_2}{2}-\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}+2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}-\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2}-\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}$$
$$+\sin^4\frac{\alpha}{2}=4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-4\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-4\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}+4\cos^4\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}$$

$$\Rightarrow\ cos^4\frac{\alpha_1}{2}+\ cos^4\frac{\alpha_2}{2}+sin^4\frac{\alpha}{2}+4\,cos^2\frac{\alpha_1}{2}cos^2\frac{\alpha_2}{2}sin^2\frac{\alpha}{2}-2\,cos^2\frac{\alpha_1}{2}cos^2\frac{\alpha_2}{2}-2\,cos^2\frac{\alpha_1}{2}sin^2\frac{\alpha}{2}-2\,cos^2\frac{\alpha_2}{2}sin^2\frac{\alpha}{2}=0$$

$$\Rightarrow\ cos^4\frac{\alpha_1}{2}+\ cos^4\frac{\alpha_2}{2}+sin^4\frac{\alpha}{2}+\ cos^2\frac{\alpha_1}{2}cos^2\frac{\alpha_2}{2}\left(4sin^2\frac{\alpha}{2}-2\right)-2sin^2\frac{\alpha}{2}\left(\ cos^2\frac{\alpha_1}{2}+\ cos^2\frac{\alpha_2}{2}\right)=0$$

Now, substituting all the corresponding values from eq(I), (II) & (III) in above expression, we have

$$\left(\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}}\right)^4+\left(\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}}\right)^4+\left(\sqrt{\frac{ab}{(a+c)(b+c)}}\right)^4+\left(\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}}\right)^2\left(\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}}\right)^2\left(4\left(\sqrt{\frac{ab}{(a+c)(b+c)}}\right)^2-2\right)-2\left(\sqrt{\frac{ab}{(a+c)(b+c)}}\right)^2\left\{\left(\sqrt{\frac{c(b+c+r)}{(b+c)(c+r)}}\right)^2+\left(\sqrt{\frac{c(a+c+r)}{(a+c)(c+r)}}\right)^2\right\}=0$$

$$\frac{c^2(b+c+r)^2}{(b+c)^2(c+r)^2}+\frac{c^2(a+c+r)^2}{(a+c)^2(c+r)^2}+\frac{a^2b^2}{(a+c)^2(b+c)^2}+\frac{c^2(a+c+r)(b+c+r)}{(a+c)(b+c)(c+r)^2}\left(\frac{4ab}{(a+c)(b+c)}-2\right)-\frac{2abc}{(a+c)(b+c)(c+r)}\left\{\frac{(b+c+r)}{(b+c)}+\frac{(a+c+r)}{(a+c)}\right\}=0$$

Now, on multiplying the above equation by $(a+c)^2(b+c)^2(c+r)^2$, we get

$$c^2(a+c)^2(b+c+r)^2+c^2(b+c)^2(a+c+r)^2+a^2b^2(c+r)^2+c^2(a+c+r)(b+c+r)(2ab-2bc-2ac-2c^2)-2abc(c+r)\{(a+c)(b+c+r)+(b+c)(a+c+r)\}=0$$

$$\Rightarrow\ c^2(a+c)^2\{r^2+2(b+c)r+(b+c)^2\}+c^2(b+c)^2\{r^2+2(a+c)r+(a+c)^2\}+a^2b^2(r^2+2cr+c^2)+c^2(2ab-2bc-2ac-2c^2)\{r^2+(a+b+2c)r+(a+c)(b+c)\}-2abc(c+r)\{(a+b+2c)r+2(a+c)(b+c)\}=0$$

$$\Rightarrow\ \{c^2(a+c)^2+c^2(b+c)^2+a^2b^2+2c^2(ab-bc-ac-c^2)\}r^2+\{2c^2(a+c)(b+c)(a+b+2c)+2a^2b^2c+2c^2(a+b+2c)(ab-bc-ac-c^2)\}r+2c^2(a+c)^2(b+c)^2+a^2b^2c^2+2c^2(a+c)(b+c)(ab-bc-ac-c^2)-2abc(a+b+2c)r^2-2abc\{2(a+c)(b+c)+c(a+b+2c)\}r-4abc^2(a+c)(b+c)=0$$

$$\Rightarrow\ \{a^2c^2+c^4+2ac^3+b^2c^2+c^4+2bc^3+a^2b^2+2abc^2-2bc^3-2ac^3-2c^4-2a^2bc-2ab^2c-4abc^2\}r^2+\{2a^2bc^2+2a^2c^3+2bc^4+2c^5+4abc^3+4ac^4+2ab^2c^2+2b^2c^3+2ac^4+2c^5+4abc^3+4bc^4+2a^2b^2c+2a^2bc^2-2abc^3-2a^2c^3-2ac^4+2ab^2c^2-2b^2c^3-2abc^3-2bc^4+4abc^3-4bc^4-4ac^4-4c^5-4a^2b^2c-6ab^2c^2-6a^2bc^2-8abc^3\}r+\{a^2b^2c^2+4a^2b^2c^2+4ab^2c^3+4a^2bc^3+4abc^4-4a^2b^2c^2-4ab^2c^3-4a^2bc^3-4abc^4\}=0$$

$$\Rightarrow\ \{a^2b^2+b^2c^2+c^2a^2-2abc(a+b+c)\}r^2-2abc(ab+bc+ca)r+a^2b^2c^2=0$$

Now, solving the above quadratic equation for the values of $r$ as follows

$$r = \frac{2abc(ab+bc+ca) \pm \sqrt{\{2abc(ab+bc+ca)\}^2 - 4\{a^2b^2c^2\}\{a^2b^2+b^2c^2+c^2a^2-2abc(a+b+c)\}}}{2\{a^2b^2+b^2c^2+c^2a^2-2abc(a+b+c)\}}$$

$$= \frac{2abc(ab+bc+ca) \pm 2abc\sqrt{4a^2bc+4ab^2c+4abc^2}}{2\{a^2b^2+b^2c^2+c^2a^2-2abc(a+b+c)\}} = \frac{abc(ab+bc+ca) \pm abc\sqrt{4abc(a+b+c)}}{(ab+bc+ca)^2 - 4abc(a+b+c)}$$

$$\Rightarrow r = abc\left(\frac{(ab+bc+ca) \pm 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - \left(2\sqrt{abc(a+b+c)}\right)^2}\right)$$

**Case 1: Taking positive sign**, we get

$$r = abc\left(\frac{(ab+bc+ca) + 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - \left(2\sqrt{abc(a+b+c)}\right)^2}\right) = abc\left(\frac{1}{(ab+bc+ca) - 2\sqrt{abc(a+b+c)}}\right)$$

$$\Rightarrow \boldsymbol{r < 0 \quad \forall\, a, b, c > 0 \;\; but\; r > 0 \;\; hence\; this\; value\; of\; radius\; r\; is\; discarded}$$

**Case 2: Taking negative sign**, we get

$$r = abc\left(\frac{(ab+bc+ca) - 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - \left(2\sqrt{abc(a+b+c)}\right)^2}\right) = abc\left(\frac{1}{(ab+bc+ca) + 2\sqrt{abc(a+b+c)}}\right)$$

$$\Rightarrow \boldsymbol{r > 0 \quad \forall\, a, b, c > 0 \;\; hence\; this\; value\; of\; radius\; r\; is\; accepted}$$

**Hence, the radius $(\boldsymbol{r})$ of inscribed circle is given as**

$$\boldsymbol{r = \frac{abc}{2\sqrt{abc(a+b+c)} + (ab+bc+ca)}} \qquad \boldsymbol{(r > 0 \quad \forall\, a, b, c > 0)}$$

Above is the **required expression to compute the radius $(\boldsymbol{r})$ of the inscribed circle which externally touches three given circles with radii $\boldsymbol{a}, \boldsymbol{b}$ & $\boldsymbol{c}$ touching each other externally**.

**3. Derivation of the radius of circumscribed circle:** Let $R$ be the radius of circumscribed circle, with centre O, is internally touched by the given circles, having centres A, B & C and radii $a, b$ & $c$, at the points M, N & P respectively. Now join the centre O to the centres A, B & C by dotted straight lines to obtain $\Delta AOB$, $\Delta BOC$ & $\Delta AOC$ & also join the centres A, B & C by dotted straight lines to obtain $\Delta ABC$ (As shown in the figure 3) Thus we have

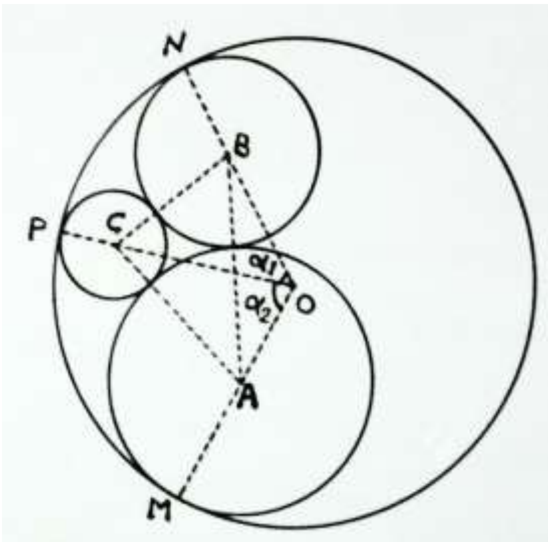


**Figure 3: The centres A, B, C & O are joined to each other by dotted straight lines to obtain $\Delta\boldsymbol{ABC}, \Delta\boldsymbol{AOB}, \Delta\boldsymbol{BOC}$ & $\Delta\boldsymbol{AOC}$**

$$AM = a, \qquad BN = b, \qquad CP = c \quad \&$$

$$OM = ON = OP = R\; (radius\; of\; circumscribed\; circle)$$

$$OA = OM - AM = R - a, \;\; OB = R - b \;\; \& \;\; OC = R - c$$

In $\Delta AOB$

$$OA = R - a, \;\; AB = a + b \;\; \& \;\; OB = R - b$$

$$semiperimeter = \frac{OA + AB + OB}{2}$$

$$s = \frac{R - a + a + b + R - b}{2} = R$$

$$\sin\frac{\angle AOB}{2} = \sqrt{\frac{(s - OA)(s - OB)}{(OA)(OB)}} = \sqrt{\frac{(R - R + a)(R - R + b)}{(R - a)(R - b)}} = \sqrt{\frac{ab}{(R - a)(R - b)}} \quad \boldsymbol{let \angle AOB = \alpha}$$

$$\Rightarrow \boldsymbol{\sin\frac{\alpha}{2}} = \sqrt{\frac{\boldsymbol{ab}}{(\boldsymbol{R - a})(\boldsymbol{R - b})}} \quad \ldots\ldots\ldots\ldots\ldots.(I)$$

Similarly, in $\Delta BOC$

$$semiperimeter = \frac{OB + BC + OC}{2}$$

$$s = \frac{R - b + b + c + R - c}{2} = R$$

$$\cos\frac{\angle BOC}{2} = \sqrt{\frac{s(s - BC)}{(OB)(OC)}} \quad \Rightarrow \boldsymbol{\cos\frac{\alpha_1}{2}} = \sqrt{\frac{\boldsymbol{R(R - b - c)}}{(\boldsymbol{R - b})(\boldsymbol{R - c})}} \quad \ldots\ldots\ldots\ldots\ldots.(II)$$

Similarly, in $\Delta AOC$

$$semiperimeter = \frac{OA + AC + OC}{2} \Rightarrow s = \frac{R - a + a + c + R - c}{2} = R$$

$$\cos\frac{\angle AOC}{2} = \sqrt{\frac{s(s - AC)}{(OA)(OC)}} \quad \Rightarrow \boldsymbol{\cos\frac{\alpha_2}{2}} = \sqrt{\frac{\boldsymbol{R(R - a - c)}}{(\boldsymbol{R - a})(\boldsymbol{R - c})}} \quad \ldots\ldots\ldots\ldots\ldots.(III)$$

Now, again in $\Delta AOB$, we have

$$\angle BOC + \angle AOC = \angle AOB \ \Rightarrow \alpha_2 + \alpha_1 = \alpha \ \ or \ \ \frac{\alpha_1}{2} + \frac{\alpha_2}{2} = \frac{\alpha}{2}$$

Now, taking **cosines** of both the sides we have

$$\cos\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right) = \cos\frac{\alpha}{2} \Rightarrow \cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2} = \cos\frac{\alpha}{2}$$

$$\Rightarrow \left(\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} - \sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\right)^2 = \left(\cos\frac{\alpha}{2}\right)^2$$

$$\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} + \sin^2\frac{\alpha_1}{2}\sin^2\frac{\alpha_2}{2} - 2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2} = \cos^2\frac{\alpha}{2}$$

$$\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} + \left(1 - \cos^2\frac{\alpha_1}{2}\right)\left(1 - \cos^2\frac{\alpha_2}{2}\right) - \cos^2\frac{\alpha}{2} = 2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}$$

$$\Rightarrow \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} + 1 - \cos^2\frac{\alpha_1}{2} - \cos^2\frac{\alpha_1}{2} + \cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - \cos^2\frac{\alpha}{2} = 2\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}$$

$$\Rightarrow 2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2} - \cos^2\frac{\alpha_1}{2} - \cos^2\frac{\alpha_1}{2} + \sin^2\frac{\alpha}{2} = 2\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}\sqrt{\left(1 - \cos^2\frac{\alpha_1}{2}\right)}\sqrt{\left(1 - \cos^2\frac{\alpha_2}{2}\right)}$$

$$\Rightarrow \left(2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-\cos^2\frac{\alpha_1}{2}-\cos^2\frac{\alpha_1}{2}+\sin^2\frac{\alpha}{2}\right)^2=4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\left(1-\cos^2\frac{\alpha_1}{2}\right)\left(1-\cos^2\frac{\alpha_2}{2}\right)$$

$$\Rightarrow 4\cos^4\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}-2\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-2\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}+2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}-2\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}$$
$$+\cos^4\frac{\alpha_1}{2}+\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2}-2\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}+\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}$$
$$+\cos^4\frac{\alpha_2}{2}-\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}+2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}-\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2}-\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}$$
$$+\sin^4\frac{\alpha}{2}=4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-4\cos^4\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-4\cos^2\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}+4\cos^4\frac{\alpha_1}{2}\cos^4\frac{\alpha_2}{2}$$

$$\Rightarrow \cos^4\frac{\alpha_1}{2}+\cos^4\frac{\alpha_2}{2}+\sin^4\frac{\alpha}{2}+4\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}-2\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}-2\cos^2\frac{\alpha_1}{2}\sin^2\frac{\alpha}{2}$$
$$-2\cos^2\frac{\alpha_2}{2}\sin^2\frac{\alpha}{2}=0$$

$$\Rightarrow \cos^4\frac{\alpha_1}{2}+\cos^4\frac{\alpha_2}{2}+\sin^4\frac{\alpha}{2}+\cos^2\frac{\alpha_1}{2}\cos^2\frac{\alpha_2}{2}\left(4\sin^2\frac{\alpha}{2}-2\right)-2\sin^2\frac{\alpha}{2}\left(\cos^2\frac{\alpha_1}{2}+\cos^2\frac{\alpha_2}{2}\right)=0$$

Now, substituting all the corresponding values from eq(I), (II) & (III) in above expression, we have

$$\left(\sqrt{\frac{R(R-b-c)}{(R-b)(R-c)}}\right)^4+\left(\sqrt{\frac{R(R-a-c)}{(R-a)(R-c)}}\right)^4+\left(\sqrt{\frac{ab}{(R-a)(R-b)}}\right)^4$$
$$+\left(\sqrt{\frac{R(R-b-c)}{(R-b)(R-c)}}\right)^2\left(\sqrt{\frac{R(R-a-c)}{(R-a)(R-c)}}\right)^2\left(4\left(\sqrt{\frac{ab}{(R-a)(R-b)}}\right)^2-2\right)$$
$$-2\left(\sqrt{\frac{ab}{(R-a)(R-b)}}\right)^2\left\{\left(\sqrt{\frac{R(R-b-c)}{(R-b)(R-c)}}\right)^2+\left(\sqrt{\frac{R(R-a-c)}{(R-a)(R-c)}}\right)^2\right\}=0$$

$$\frac{R^2(R-b-c)^2}{(R-b)^2(R-c)^2}+\frac{R^2(R-a-c)^2}{(R-a)^2(R-c)^2}+\frac{a^2b^2}{(R-a)^2(R-b)^2}$$
$$+\frac{R^2(R-b-c)(R-a-c)}{(R-a)(R-b)(R-c)^2}\left(\frac{4ab}{(R-a)(R-b)}-2\right)$$
$$-\frac{2abR}{(R-a)(R-b)(R-c)}\left\{\frac{(R-b-c)}{(R-b)}+\frac{(R-a-c)}{(R-a)}\right\}=0$$

Now, on multiplying the above equation by $(R-a)^2(R-b)^2(R-c)^2$, we get

$$R^2(R-a)^2(R-b-c)^2+R^2(R-b)^2(R-a-c)^2+a^2b^2(R-c)^2$$
$$+R^2(R-b-c)(R-a-c)\{2ab-2R^2+2(a+b)R\}$$
$$-2abR(R-c)\{(R-a)(R-b-c)+(R-b)(R-a-c)\}=0$$

$$\Rightarrow R^2(R-a)^2\{R^2-2(b+c)R+(b+c)^2\}+R^2(R-b)^2\{R^2-2(a+c)R+(a+c)^2\}$$
$$+a^2b^2(R^2-2cR+c^2)$$
$$+R^2\{R^2-(a+b+2c)R+(a+c)(b+c)\}\{2ab-2R^2+2(a+b)R\}$$
$$-2abR(R-c)\{2R^2-2(a+b+c)R+a(b+c)+b(a+c)\}=0$$

$$\Rightarrow \{R^4-2(b+c)R^3+(b+c)^2R^2\}(R^2+a^2-2aR)+\{R^4-2(a+c)R^3+(a+c)^2R^2\}(R^2+b^2-2bR)$$
$$+a^2b^2R^2-2a^2b^2cR+a^2b^2c^2$$
$$+\{R^4-(a+b+2c)R^3+(a+c)(b+c)R^2\}\{2ab-2R^2+2(a+b)R\}$$
$$+(2abcR-2abR^2)\{2R^2-2(a+b+c)R+a(b+c)+b(a+c)\}=0$$

$$\Rightarrow R^6 - 2(b+c)R^5 + (b+c)^2R^4 + a^2R^4 - 2a^2(b+c)R^3 + a^2(b+c)^2R^2 - 2aR^5 + 4a(b+c)R^4 - 2a(b+c)^2R^3 + R^6 - 2(a+c)R^5 + (a+c)^2R^4 + b^2R^4 - 2b^2(a+c)R^3 + b^2(a+c)^2R^2 - 2bR^5 + 4b(a+c)R^4 - 2b(a+c)^2R^3 + a^2b^2R^2 - 2a^2b^2cR + a^2b^2c^2 + 2abR^4 - 2ab(a+b+2c)R^3 + 2ab(a+c)(b+c)R^2 - 2R^6 + 2(a+b+2c)R^5 - 2(a+c)(b+c)R^4 - 4abc(a+b+c)R^2 + 4ab(a+b+c)R^3 + 2abc(2ab+ac+bc)R - 2ab(2ab+ac+bc)R^2 = 0$$

$$\Rightarrow \{a^2b^2 + b^2c^2 + c^2a^2 - 2abc(a+b+c)\}R^2 + 2abc(ab+bc+ca)R + a^2b^2c^2 = 0$$

Now, solving the above quadratic equation for the values of $R$ as follows

$$R = \frac{-2abc(ab+bc+ca) \pm \sqrt{\{-2abc(ab+bc+ca)\}^2 - 4\{a^2b^2c^2\}\{a^2b^2+b^2c^2+c^2a^2-2abc(a+b+c)\}}}{2\{a^2b^2+b^2c^2+c^2a^2-2abc(a+b+c)\}}$$

$$= \frac{-2abc(ab+bc+ca) \pm 2abc\sqrt{4a^2bc+4ab^2c+4abc^2}}{2\{a^2b^2+b^2c^2+c^2a^2-2abc(a+b+c)\}} = \frac{-abc(ab+bc+ca) \pm abc\sqrt{4abc(a+b+c)}}{(ab+bc+ca)^2 - 4abc(a+b+c)}$$

$$\Rightarrow R = abc\left(\frac{-(ab+bc+ca) \pm 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - \left(2\sqrt{abc(a+b+c)}\right)^2}\right)$$

**Case 1: Taking positive sign**, we get

$$R = abc\left(\frac{-(ab+bc+ca) + 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - \left(2\sqrt{abc(a+b+c)}\right)^2}\right) = -abc\left(\frac{1}{(ab+bc+ca) + 2\sqrt{abc(a+b+c)}}\right)$$

$$\Rightarrow \boldsymbol{R < 0 \quad \forall\, a, b, c > 0 \;\; but\; R > 0 \;\; hence\ this\ value\ of\ radius\ R\ is\ discarded}$$

**Case 2: Taking negative sign**, we get

$$R = abc\left(\frac{-(ab+bc+ca) - 2\sqrt{abc(a+b+c)}}{(ab+bc+ca)^2 - \left(2\sqrt{abc(a+b+c)}\right)^2}\right) = -abc\left(\frac{1}{(ab+bc+ca) - 2\sqrt{abc(a+b+c)}}\right)$$

$$= abc\left(\frac{1}{2\sqrt{abc(a+b+c)} - (ab+bc+ca)}\right)$$

$$\Rightarrow \boldsymbol{R > 0 \quad \forall\, a, b, c > 0 \;\; hence\ this\ value\ of\ radius\ R\ is\ accepted}$$

**Hence, the radius $(\boldsymbol{R})$ of circumscribed circle is given as**

$$\boldsymbol{R = \frac{abc}{2\sqrt{abc(a+b+c)} - (ab+bc+ca)}} \qquad (\boldsymbol{R > 0 \quad \forall\, a, b, c > 0})$$

Above is the **required expression to compute the radius $(\boldsymbol{R})$ of the circumscribed circle which is internally touched by three given circles with radii $\boldsymbol{a}, \boldsymbol{b}$ & $\boldsymbol{c}$ touching each other externally**.

**NOTE:** The circumscribed circle will exist for three given radii $a, b$ & $c$ $(a \geq b \geq c > 0)$ if & only if the following inequality is satisfied

$$c > \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$$

For any other value of radius $c$ (of smallest circle) not satisfying the above inequality, the circumscribed circle will not exist i.e. there will be no circle which circumscribes & internally touches three externally touching circles if the above inequality fails to hold good.

**Special case:** If three circles of equal radius $a$ are touching each other externally then the **radii** $\boldsymbol{r}$ & $\boldsymbol{R}$ **of inscribed & circumscribed circles** respectively are obtained by setting $a = b = c = a$ in the above expressions as follows

$$\Rightarrow \ \boldsymbol{r} = \frac{abc}{2\sqrt{abc(a+b+c)} + (ab+bc+ca)} = \frac{a^3}{2\sqrt{a^3(a+a+a)} + (a^2+a^2+a^2)} = \frac{a^3}{2\sqrt{3}a^2 + 3a^2}$$

$$= \frac{a}{2\sqrt{3}+3} = \frac{a\left(2\sqrt{3}-3\right)}{\left(2\sqrt{3}+3\right)\left(2\sqrt{3}-3\right)} = \frac{a\left(2\sqrt{3}-3\right)}{3} = \boldsymbol{a}\left(\frac{\mathbf{2}}{\sqrt{\mathbf{3}}} - \mathbf{1}\right) \approx \mathbf{0.154700538}\boldsymbol{a}$$

$$\Rightarrow \ \boldsymbol{R} = \frac{abc}{2\sqrt{abc(a+b+c)} - (ab+bc+ca)} = \frac{a^3}{2\sqrt{a^3(a+a+a)} - (a^2+a^2+a^2)} = \frac{a^3}{2\sqrt{3}a^2 - 3a^2}$$

$$= \frac{a}{2\sqrt{3}-3} = \frac{a\left(2\sqrt{3}+3\right)}{\left(2\sqrt{3}-3\right)\left(2\sqrt{3}+3\right)} = \frac{a\left(2\sqrt{3}+3\right)}{3} = \boldsymbol{a}\left(\frac{\mathbf{2}}{\sqrt{\mathbf{3}}} + \mathbf{1}\right) \approx \mathbf{2.154700538}\boldsymbol{a}$$

**4. Derivation of the radius of inscribed circle:** Let $r$ be the radius of inscribed circle, with centre C, externally touching two given externally touching circles, having centres A & B and radii $a$ & $b$ respectively, and their common tangent MN. Now join the centres A, B & C to each other as well as to the points of tangency M, N & P respectively by dotted straight lines. Draw the perpendicular AT from the centre A to the line BN. Also draw a line passing through the centre C & parallel to the tangent MN which intersects the lines AM & BN at the points Q & S respectively. (As shown in the figure 4) Thus we have

$$AM = a, \qquad BN = b, \qquad CP = r = ?$$

In right $\Delta ATB$

$$AB = a+b \ \ \& \ \ BT = BN - TN = BN - AM = b - a$$

$$\Rightarrow \ AT = \sqrt{(AB)^2 - (BT)^2}$$

$$= \sqrt{(a+b)^2 - (b-a)^2} = \sqrt{4ab} = 2\sqrt{ab}$$

$$\therefore \ \boldsymbol{AT} = \boldsymbol{QS} = \boldsymbol{MN} = \mathbf{2}\sqrt{\boldsymbol{ab}} \quad \ldots\ldots\ldots\ldots (I)$$

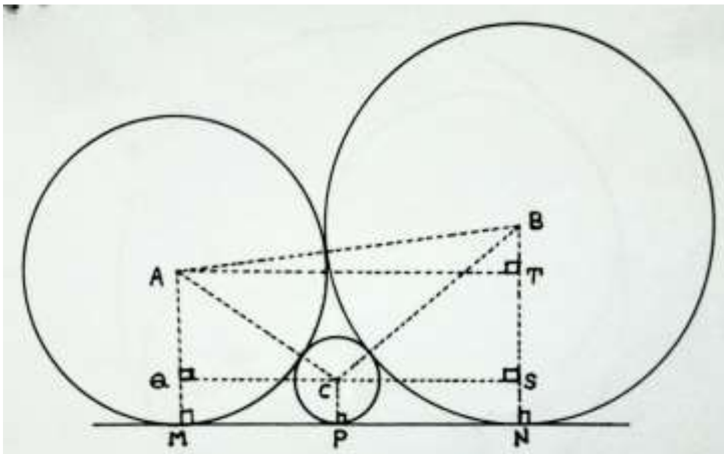


**Figure 4: A small circle with centre C is externally touching two given externally touching circles with centres A & B and their common tangent MN**

In right $\Delta AQC$

$$AC = a + r \ \ \& \ \ AQ = AM - QM = AM - CP = a - r$$

$$\Rightarrow \ QC = \sqrt{(AC)^2 - (AQ)^2}$$

$$= \sqrt{(a+r)^2 - (a-r)^2} = \sqrt{4ar} = 2\sqrt{ar} \ \therefore \ \boldsymbol{QC} = \boldsymbol{MP} = \mathbf{2}\sqrt{\boldsymbol{ar}} \quad \ldots\ldots\ldots\ldots (II)$$

In right $\Delta BSC$

$$BC = b + r \;\; \& \;\; BS = BN - SN = BN - CP = b - r$$

$$\Rightarrow CS = \sqrt{(BC)^2 - (BS)^2} = \sqrt{(b+r)^2 - (b-r)^2} = \sqrt{4br} = 2\sqrt{br} \;\therefore\; \boldsymbol{CS = PN = 2\sqrt{br}} \quad \dots\dots\dots\dots (III)$$

From the above figure 4, it is obvious that $\boldsymbol{MP} + \boldsymbol{PN} = \boldsymbol{MN}$ now, substituting the corresponding values, we get

$$2\sqrt{ar} + 2\sqrt{br} = 2\sqrt{ab} \quad \Rightarrow \sqrt{r}\left(\sqrt{a} + \sqrt{b}\right) = \sqrt{ab} \quad \Rightarrow \sqrt{r} = \frac{\sqrt{ab}}{\left(\sqrt{a} + \sqrt{b}\right)}$$

$$\Rightarrow r = \left(\frac{\sqrt{ab}}{\left(\sqrt{a} + \sqrt{b}\right)}\right)^2 = \frac{ab}{a + b + 2\sqrt{ab}}$$

$$\therefore \;\; \boldsymbol{r = \frac{ab}{a + b + 2\sqrt{ab}} = \frac{ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}} \qquad \boldsymbol{(r > 0 \;\; \forall\, a, b > 0)}$$

Above is the **required expression to compute the radius $(r)$ of the inscribed circle which externally touches two given circles with radii $a$ & $b$ & their common tangent**.

**Special case:** If two circles of equal radius $a$ are touching each other externally then the **radius $r$ of inscribed circle** externally touching them as well as their common tangent, is obtained by setting $a = b = a$ in the above expressions as follows

$$r = \frac{ab}{a + b + 2\sqrt{ab}} = \frac{a^2}{a + a + 2\sqrt{a^2}} = \frac{a^2}{4a} = \frac{a}{4} \;\Rightarrow\; \boldsymbol{r = \frac{a}{4}}$$

## 5. Relationship of the radii of three externally touching circles enclosed in a smallest rectangle:

Consider any three externally touching circles with the centres A, B & C and their radii $a, b$ & $c$ $(\forall a > b \geq c)$ respectively enclosed in a smallest rectangle PQRS. (See the figure 5)

Now, draw the perpendiculars AD, AF & AH from the centre A of the biggest circle to the sides PQ, RS & QR respectively. Also draw the perpendiculars CE & CM from the centre C to the straight lines PQ & DF respectively and the perpendiculars BG & BN from the centre B to the straight lines RS & DF respectively. Then join the centres A, B & C to each other by the (dotted) straight lines to obtain $\Delta ABC$. Now, we have

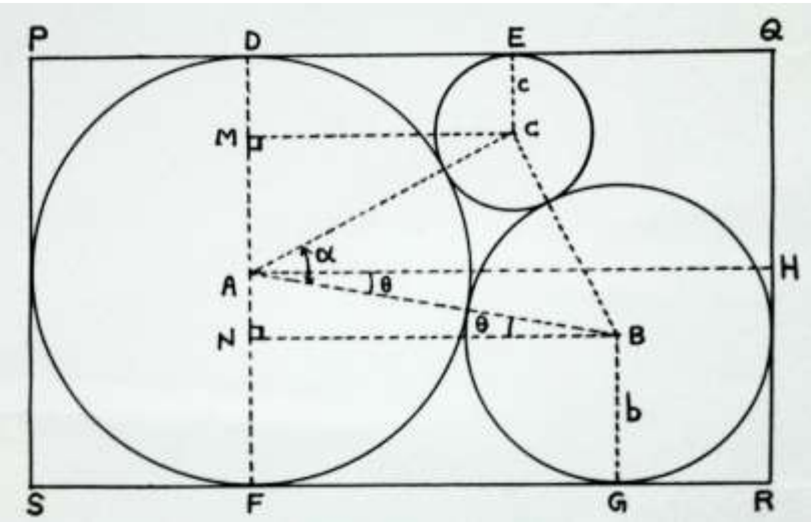


**Figure 5: Three externally touching circles with their centres A, B & C and radii $a, b$ & $c$ $(\forall \; a > b \geq c)$ respectively are enclosed in a smallest rectangle PQRS.**

$$AD = AF = a, \;\; BG = b \;\; \& \;\; CE = c \quad (\forall \, b, c < a)$$

$$AB = a + b, \;\; BC = b + c \;\&\; AC = a + c$$

Now, applying cosine rule in right $\Delta ABC$

$$cos\angle BAC = \frac{(AB)^2 + (AC)^2 - (BC)^2}{2(AB)(AC)} \;\; \Rightarrow \; cos\alpha = \frac{(a+b)^2 + (a+c)^2 - (b+c)^2}{2(a+b)(a+c)}$$

$$cos\alpha = \frac{a^2 + b^2 + 2ab + a^2 + c^2 + 2ac - b^2 - c^2 - 2bc}{2(a+b)(a+c)} = \frac{a^2 + ab + ac - bc}{(a+b)(a+c)} = \frac{a(a+b) + c(a-b)}{(a+b)(a+c)}$$

$$\boldsymbol{cos\alpha = \frac{a(a+b) + c(a-b)}{(a+b)(a+c)}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(I)$$

$$\Rightarrow \; sin\alpha = \sqrt{1 - cos^2\alpha} = \sqrt{1 - \left(\frac{a(a+b) + c(a-b)}{(a+b)(a+c)}\right)^2}$$

$$= \frac{\sqrt{a^2(a+b)^2 + c^2(a+b)^2 + 2ac(a+b)^2 - a^2(a+b)^2 - c^2(a-b)^2 - 2ac(a^2 - b^2)}}{(a+b)(a+c)}$$

$$\boldsymbol{sin\alpha = \frac{\sqrt{4abc^2 + 4abc(a+b)}}{(a+b)(a+c)}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(II)$$

In right $\Delta ANB$

$$\sin\angle ABN = \frac{AN}{AB} = \frac{AF - NF}{AB} = \frac{AF - BG}{AB}$$

$$\Rightarrow \; \boldsymbol{sin\theta = \frac{a-b}{a+b}} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(III)$$

$$\cos\angle ABN = \frac{BN}{AB} = \frac{\sqrt{(AB)^2 - (AN)^2}}{AB} = \frac{\sqrt{(a+b)^2 - (a-b)^2}}{a+b} = \frac{\sqrt{4ab}}{a+b}$$

$$\Rightarrow \; \boldsymbol{cos\theta = \frac{2\sqrt{ab}}{a+b}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(IV)$$

In right $\Delta AMC$

$$sin\angle ACM = \frac{AM}{AC} = \frac{AD - MD}{AC} = \frac{AD - CE}{AC} \quad \Rightarrow \; sin(\alpha - \theta) = \frac{a-c}{a+c}$$

$$\Rightarrow \;\; (a+c)sin(\alpha - \theta) = a - c \;\;\; or \;\;\; (a+c)(sin\alpha cos\theta - cos\alpha sin\theta) = a - c$$

Now, by substituting the corresponding values from the eq(I), (II), (III) & (IV) in the above expression, we get

$$(a+c)\left(\frac{\sqrt{4abc^2 + 4abc(a+b)}}{(a+b)(a+c)} \times \frac{2\sqrt{ab}}{a+b} - \frac{a(a+b) + c(a-b)}{(a+b)(a+c)} \times \frac{a-b}{a+b}\right) = a - c$$

$$\Rightarrow \; (a+c)\left(\frac{4ab\sqrt{c^2 + c(a+b)}}{(a+b)^2(a+c)} - \frac{a(a+b)(a-b) + c(a-b)^2}{(a+b)^2(a+c)}\right) = a - c$$

$$\Rightarrow \; 4ab\sqrt{c^2 + c(a+b)} - a(a^2 - b^2) - c(a-b)^2 = (a-c)(a+b)^2$$

$$\Rightarrow \; 4ab\sqrt{c^2 + c(a+b)} = a(a+b)^2 - c(a+b)^2 + a(a^2 - b^2) + c(a-b)^2$$

$$\Rightarrow \; 4ab\sqrt{c^2 + c(a+b)} = a\{(a+b)^2 + a^2 - b^2\} - c\{(a+b)^2 - (a-b)^2\}$$

$$\Rightarrow\ 4ab\sqrt{c^2+c(a+b)} = a(2a^2+2ab)-c(4ab)$$

$$\Rightarrow\ 2b\sqrt{c^2+c(a+b)} = a(a+b)-2bc$$

Now, taking the square on both the sides, we get

$$\left(2b\sqrt{c^2+c(a+b)}\right)^2 = (a(a+b)-2bc)^2$$

$$\Rightarrow\ 4b^2\left(c^2+c(a+b)\right) = a^2(a+b)^2+4b^2c^2-4abc(a+b)$$

$$\Rightarrow\ 4b^2c^2+4b^2c(a+b) = a^2(a+b)^2+4b^2c^2-4abc(a+b)$$

$$\Rightarrow\ 4b^2c(a+b)+4abc(a+b) = a^2(a+b)^2$$

$$\Rightarrow\ 4bc(a+b)(b+a) = a^2(a+b)^2\ \ or\ \ 4bc(a+b)^2 = a^2(a+b)^2\ \Rightarrow\ 4bc = a^2$$

$$\Rightarrow\ \boldsymbol{a^2 = 4bc\ \ or\ \ a = 2\sqrt{bc}\qquad \forall\ a > b \geq c}$$

**Above relation is very important for computing any of the radii $a, b\ \&\ c$ if other two are known for three externally touching circles enclosed in a smallest rectangle.**

**Dimensions of the smallest enclosing rectangle:** The length $L$ & width $B$ of the smallest rectangle PQRS enclosing three externally circles touching circles are calculated as follows (see the figure 5 above)

$$Length,\ \ L = PQ = RS = SF+FG+GR = a+2\sqrt{ab}+b = a+b+2\sqrt{ab} = \left(\sqrt{a}+\sqrt{b}\right)^2$$

$$Width,\ \ B = PS = QR = DM = 2a$$

$$\therefore\ \boldsymbol{Length, L = \left(\sqrt{a}+\sqrt{b}\right)^2\ \ \&\ Width, B = 2a\qquad \forall\ a^2 = 4bc\ \ \&\ \ a > b \geq c}$$

**Thus, above expressions can be used to compute the dimensions of the smallest rectangle enclosing three externally touching circles having radii $a, b\ \&\ c\ \ (a > b \geq c)$.**

**6. Length of common chord of two intersecting circles:** Consider two circles with centres $O_1\ \&\ O_2$ and radii $r_1\ \&\ r_2$ respectively, at a distance $d$ between their centres, intersecting each other at the points A & B (As shown in the figure 6). Join the centres $O_1\ \&\ O_2$ to the point A. The line $O_1O_2$ bisects the common chord AB perpendicularly at the point M. Let $AM = x$ then the length of common chord $AB = 2x$. Now

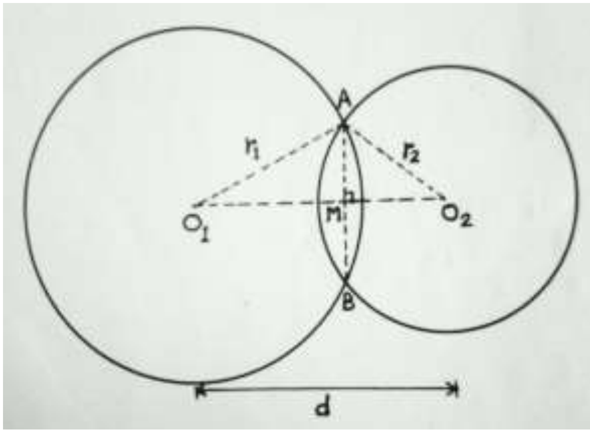


**Figure 6: Two circles with the centres $O_1\ \&\ O_2$ and radii $r_1\ \&\ r_2$ respectively at a distance d between their centres, intersecting each other at the points A & B**

In right triangle $\Delta AMO_1$,

$$O_1M = \sqrt{(O_1A)^2-(AM)^2} = \sqrt{{r_1}^2-x^2}$$

Similarly, In right triangle $\Delta AMO_2$,

$$MO_2 = \sqrt{(O_2A)^2-(AM)^2} = \sqrt{{r_2}^2-x^2}$$

Now,

$$O_1O_2 = O_1M + MO_2$$

Substituting the corresponding values, we get

$$d = \sqrt{{r_1}^2 - x^2} + \sqrt{{r_2}^2 - x^2}$$

Taking squares on both the sides,

$$d^2 = \left(\sqrt{{r_1}^2 - x^2} + \sqrt{{r_2}^2 - x^2}\right)^2$$

$${r_1}^2 - x^2 + {r_2}^2 - x^2 + 2\sqrt{({r_1}^2 - x^2)({r_2}^2 - x^2)} = d^2$$

$$2\sqrt{({r_1}^2 - x^2)({r_2}^2 - x^2)} = 2x^2 + d^2 - {r_1}^2 - {r_2}^2$$

$$4({r_1}^2 - x^2)({r_2}^2 - x^2) = (2x^2 + d^2 - {r_1}^2 - {r_2}^2)^2$$

$$4{r_1}^2{r_2}^2 - 4({r_1}^2 + {r_2}^2)x^2 + 4x^4 = 4x^4 + (d^2 - {r_1}^2 - {r_2}^2)^2 + 4(d^2 - {r_1}^2 - {r_2}^2)x^2$$

$$4d^2x^2 = 4{r_1}^2{r_2}^2 - (d^2 - {r_1}^2 - {r_2}^2)^2$$

$$4x^2 = \frac{(2r_1r_2)^2 - (d^2 - {r_1}^2 - {r_2}^2)^2}{d^2}$$

$$4x^2 = \frac{(2r_1r_2 + d^2 - {r_1}^2 - {r_2}^2)(2r_1r_2 - d^2 + {r_1}^2 + {r_2}^2)}{d^2}$$

$$4x^2 = \frac{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}{d^2}$$

$$2x = \sqrt{\frac{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}{d^2}}$$

$$\Rightarrow AB = \frac{\sqrt{(d^2 - (r_1 - r_2)^2)((r_1 + r_2)^2 - d^2)}}{d}$$

Hence, the length of the common chord of two intersecting circles with radii $r_1$ & $r_2$ at a distance $d$ between their centres is

$$\textbf{Length of common chord, L} = \frac{\sqrt{\{d^2 - (r_1 - r_2)^2\}\{(r_1 + r_2)^2 - d^2\}}}{d} \qquad \forall \ \ |r_1 - r_2| \le d \le r_1 + r_2$$

**Special case**: If $r_1 \neq r_2$ then the **maximum length of common chord** of two intersecting circles

$= 2 \times min(r_1, r_2) =$ **diameter of smaller circle** at a central distance $\boldsymbol{d} = \sqrt{|{r_1}^2 - {r_2}^2|}$

**Angles of intersection of two intersecting circles:** Let $\angle AO_1M = \theta_1$ & $\angle AO_2M = \theta_2$ (See above fig 6).

In right $\Delta AMO_1$ , we have

$$\sin\angle AO_1M = \frac{AM}{AO_1} \quad \Rightarrow \quad \sin\theta_1 = \frac{L/2}{r_1} = \frac{L}{2r_1}$$

$$\Rightarrow \boldsymbol{\theta_1} = \sin^{-1}\left(\frac{L}{2r_1}\right) = \textbf{semi aperture angle subtended by common chord AB at centre } O_1$$

Similarly, in right $\Delta AMO_2$ ,

$$\angle AO_2M = \theta_2 = \sin^{-1}\left(\frac{L}{2r_2}\right) = \textbf{semi aperture angle subtended by common chord AB at centre } O_2$$

Now, it can be easily proved that one of the two supplementary angles of intersection ($\theta$) is given as the sum of above two semi-aperture angles $\theta_1$ and $\theta_2$ subtended by common chord AB at the centres $O_1$ & $O_2$ of two intersecting circles (see above fig. 6)

$$\theta = \theta_1 + \theta_2 = \sin^{-1}\left(\frac{L}{2r_1}\right) + \sin^{-1}\left(\frac{L}{2r_2}\right)$$

Hence, both the **supplementary angles of intersection of two intersecting circles** are given as follows

$$\boldsymbol{\theta = \sin^{-1}\left(\frac{L}{2r_1}\right) + \sin^{-1}\left(\frac{L}{2r_2}\right) \quad \& \quad \pi - \theta}$$

$$Where, \ L = \frac{\sqrt{\{d^2-(r_1-r_2)^2\}\{(r_1+r_2)^2-d^2\}}}{d} \qquad \forall \ \boldsymbol{|r_1 - r_2| \le d \le r_1 + r_2}$$

**Area of intersection (A) of two intersecting circles:** As we have computed above, $2\theta_1$ is the angle of aperture subtended by common chord AB at the centre $O_1$ of circle with a radius $r_1$ hence the area ($A_1$) of segment of corresponding circle is give as (Refer to fig.6 above)

$$A_1 = \text{Area of sector } O_1AB - \text{area of isosceles } \Delta O_1AB$$

$$= \frac{1}{2}(2\theta_1){r_1}^2 - \frac{1}{2}(r_1 \times r_1)\sin 2\theta_1$$

$$= \frac{1}{2}(2\theta_1 - \sin 2\theta_1){r_1}^2$$

$$= \frac{1}{2}(2\theta_1 - 2\sin\theta_1\cos\theta_1){r_1}^2$$

$$= (\theta_1 - \sin\theta_1\cos\theta_1){r_1}^2$$

Similarly, the area ($A_2$) of segment of circle with a radius $r_2$ & aperture angle $2\theta_2$ subtended by common chord AB at the centre $O_2$ , is given as follows

$$A_2 = (\theta_2 - \sin\theta_2\cos\theta_2){r_2}^2$$

Now, the area of intersection (A) of two intersecting circles will be equal to the sum of areas $A_1$ & $A_2$ of segments as computed above

$$A = A_1 + A_2 = (\theta_1 - \sin\theta_1\cos\theta_1){r_1}^2 + (\theta_2 - \sin\theta_2\cos\theta_2){r_2}^2$$

Hence, the **area (A) of intersection of any two intersecting circles of radii $r_1$ & $r_2$ separated by a central distance $d$** is given as

$$\boldsymbol{A = (\theta_1 - \sin\theta_1\cos\theta_1){r_1}^2 + (\theta_2 - \sin\theta_2\cos\theta_2){r_2}^2}$$

$$Where, \ \theta_1 = \sin^{-1}\left(\frac{L}{2r_1}\right), \ \ \theta_2 = \sin^{-1}\left(\frac{L}{2r_2}\right) \ \& \ L = \frac{\sqrt{\{d^2-(r_1-r_2)^2\}\{(r_1+r_2)^2-d^2\}}}{d}$$

$$\forall \ \boldsymbol{|r_1 - r_2| \le d \le r_1 + r_2}$$

**Conclusion:** *All the articles above have been derived by using **simple geometry & trigonometry**. All above articles (formula) are very practical & simple to apply in case studies & practical applications of 2-D Geometry. Although above results are also valid in case of three spheres touching one another externally in 3-D geometry.*

**Note:** Above articles had been derived *& illustrated by **Mr H.C. Rajpoot** (**B Tech, Mechanical Engineering**)*

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