$Mathematical\ Analysis\ of\ Uniform\ Polyhedra$

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Introduction: Here, we are to analyse a **uniform polyhedron** having **2 congruent regular n-gonal faces**, **2n congruent trapezoidal faces, 5n edges** & **3n vertices lying on a spherical surface** with a certain radius. Each of 2n trapezoidal faces has three equal sides, two equal **acute** angles each $\boldsymbol{\alpha}$ & two equal **obtuse** angles each $\boldsymbol{\beta}$. (See the figure 1 showing a uniform tetradecahedron). **The condition, of all 3n vertices lying on a spherical surface**, **governs** & **correlates all the parameters of a uniform polyhedron** such as **solid angle** subtended by each face at the centre, **normal distance** of each face from the centre, **outer (circumscribed) radius**, **inner (inscribed) radius**, **mean radius**, **surface area**, **volume** etc. If the length of one of two unequal edges is known then all the dimensions of a uniform polyhedron can be easily determined. It is to noted that if the edge length of regular n-gonal face is known then the analysis becomes very easy. We would derive a **mathematical relation** of the side length a of regular n-gonal faces & the radius R of the spherical surface passing through all 3n vertices. Thus, all the dimensions of a uniform polyhedron can be easily determined only in terms of edge length a & plane angles, solid angles of each face can also be determined easily.

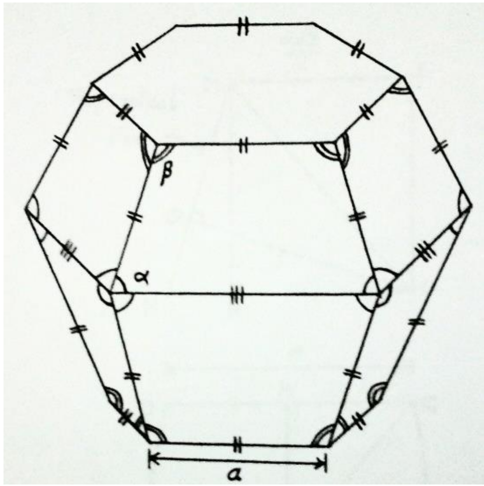

Figure 1: A uniform tetradecahedron has 2 congruent regular hexagonal faces each of edge length $\boldsymbol{a}$ & 12 congruent trapezoidal faces. All its 18 vertices eventually & exactly lie on a spherical surface with a certain radius.

$$\textbf{total no. of faces} = \mathbf{2n+2},\ \ \textbf{no. of regular polygonal faces} = \mathbf{2},$$
$$\textbf{no. of trapezoidal faces} = \mathbf{2n},\ \ \textbf{no. of edges} = \mathbf{5n}\ \ \&$$

$$\textbf{no. of vertices} = \mathbf{3n}$$

Analysis of Uniform Polyhedron: For ease of calculations & understanding, let there be a uniform polyhedron, with the centre O, having 2 congruent **regular n-gonal faces** each with **edge length $\boldsymbol{a}$** & 2n congruent **trapezoidal faces each with three equal sides each $\boldsymbol{a}$** and all its 3n vertices lying on a spherical surface with a **radius $\boldsymbol{R_o}$**. Now consider any of 2n congruent trapezoidal faces say ABCD ($AD = BC = CD = a$) & join the vertices A & D to the centre O. (See the figure 2). Join the centre E of the top regular n-gonal face to the centre O & to the vertex D. Draw a perpendicular DF from the vertex D to the line AO, perpendicular EM from the centre E to the side CD, perpendicular ON from the centre O to the side AB & then join the mid-points M & N of the sides CD & AB respectively in order to obtain trapeziums ADEO & OEMN (See the figure 3 & 4 below). Now we have,

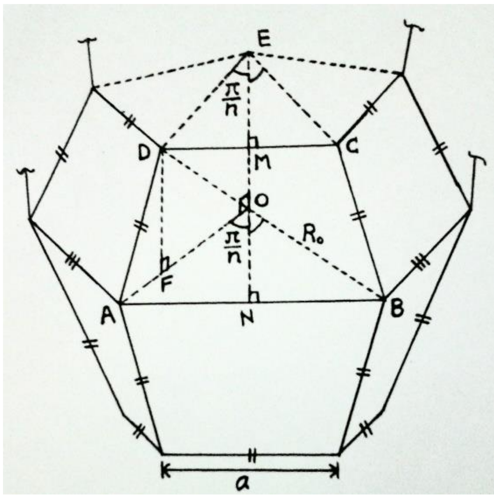

Figure 2: ABCD is one of 2n congruent trapezoidal faces with $\boldsymbol{AD = BC = CD = a}$. $\boldsymbol{\Delta CED}$ & $\boldsymbol{\Delta AOB}$ are isosceles triangles. ADEO & OEMN are trapeziums.

$$OA = OB = OD = R_o\,,\ \ AD = BC = CD = a,\ \ \angle CED = \angle AOB = \frac{2\pi}{n}$$

Hence, in isosceles triangles ΔCED & ΔAOB , we have

$$\boldsymbol{EC = ED}\ \&\ \boldsymbol{CD = a}\ \ \&\ \ \boldsymbol{OA = OB = R_o}$$

In right ΔEMD

$$\sin\angle DEM = \frac{DM}{ED} \Rightarrow \sin\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{ED} \Rightarrow \boldsymbol{ED = \frac{a}{2}cosec\frac{\pi}{n} = EC = OF}$$

$$\tan\angle DEM = \frac{DM}{EM} \Rightarrow \tan\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{EM} \Rightarrow \boldsymbol{EM = \frac{a}{2}\cot\frac{\pi}{n} = OH}$$

In right ΔANO (figure 2)

$$\sin\angle AON = \frac{AN}{OA} \Rightarrow \sin\frac{\pi}{n} = \frac{AN}{R_o} \Rightarrow \boldsymbol{AN = R_o \sin\frac{\pi}{n} = NB}$$

$$\cos\angle AON = \frac{ON}{OA} \Rightarrow \cos\frac{\pi}{n} = \frac{ON}{R_o} \Rightarrow \boldsymbol{ON = R_o \cos\frac{\pi}{n}}$$

In right ΔOED (figure 3)

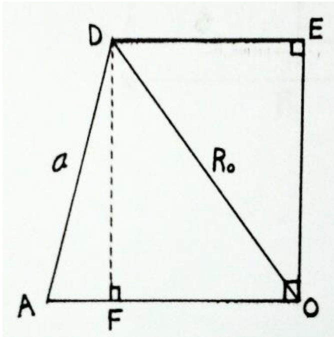

Figure 3: Trapezium ADEO with $\boldsymbol{AD = a,\ OA = OD = R_o}$ & $\boldsymbol{DF = EO}$. The lines DE & AO are parallel.

$$EO = \sqrt{(OD)^2 - (DE)^2} = \sqrt{R_o^{\ 2} - \left(\frac{a}{2}cosec\frac{\pi}{n}\right)^2}$$

$$\Rightarrow \boldsymbol{EO = DF = MH = \frac{1}{2}\sqrt{4R_o^{\ 2} - a^2cosec^2\frac{\pi}{n}}}$$

In right ΔAFD (figure 3)

$$\Rightarrow (AD)^2 = (AF)^2 + (DF)^2 = (OA - OF)^2 + (DF)^2$$

$$\Rightarrow a^2 = \left(R_o - \frac{a}{2}cosec\frac{\pi}{n}\right)^2 + \left(\frac{1}{2}\sqrt{4R_o^{\ 2} - a^2cosec^2\frac{\pi}{n}}\right)^2$$

$$\Rightarrow a^2 = R_o^{\ 2} + \frac{a^2}{4}cosec^2\frac{\pi}{n} - aR_o cosec\frac{\pi}{n} + R_o^{\ 2} - \frac{a^2}{4}cosec^2\frac{\pi}{n}$$

$$2R_o^{\ 2} - aR_o cosec\frac{\pi}{n} - a^2 = 0$$

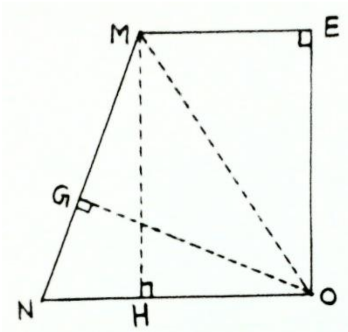

Figure 4: Trapezium OEMN with $\boldsymbol{EM = OH}$ & $\boldsymbol{EO = MH}$. The lines ME & NO are parallel.

$$\Rightarrow R_o = \frac{a cosec\frac{\pi}{n} \pm \sqrt{\left(-a cosec\frac{\pi}{n}\right)^2 + 8a^2}}{4}$$

$$= \frac{a cosec\frac{\pi}{n} \pm a\sqrt{8 + cosec^2\frac{\pi}{n}}}{4} = \frac{a}{4}\left(cosec\frac{\pi}{n} \pm \sqrt{8 + cosec^2\frac{\pi}{n}}\right)$$

But, $\boldsymbol{R_o > a > 0}$ by taking positive sign, we get

$$\therefore \quad \boldsymbol{R_o = \frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (I)$$

Now, draw a perpendicular OG from the centre O to the trapezoidal face ABCD, perpendicular MH from the mid-point M of the side CD to the line ON. Thus in **trapezium OEMN** (See the figure 4), we have

$$MH = EO = \frac{1}{2}\sqrt{4{R_o}^2 - a^2 cosec^2\frac{\pi}{n}} = \sqrt{\left(\frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\right)^2 - \frac{a^2}{4}cosec^2\frac{\pi}{n}}$$

$$= a\sqrt{\frac{1}{16}cosec^2\frac{\pi}{n} + \frac{1}{16}\left(8 + cosec^2\frac{\pi}{n}\right) + \frac{1}{8}\left(cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}\right) - \frac{1}{4}cosec^2\frac{\pi}{n}}$$

$$= a\sqrt{\frac{1}{16}cosec^2\frac{\pi}{n} + \frac{1}{2} + \frac{1}{16}cosec^2\frac{\pi}{n} + \frac{1}{8}cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - \frac{1}{4}cosec^2\frac{\pi}{n}}$$

$$= a\sqrt{\frac{1}{2} + \frac{1}{8}cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - \frac{1}{8}cosec^2\frac{\pi}{n}} = a\sqrt{\frac{4 + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - cosec^2\frac{\pi}{n}}{8}}$$

$$\therefore \boldsymbol{MH = EO = \frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (II)$$

$$\boldsymbol{NH} = ON - OH = ON - EM = R_o cos\frac{\pi}{n} - \frac{a}{2}cot\frac{\pi}{n} \quad (See\ figure\ 2\ \&\ 4)$$

$$= \frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)cos\frac{\pi}{n} - \frac{a}{2}cot\frac{\pi}{n} \qquad (setting\ the\ value\ of\ R_o\ from\ eq(I))$$

$$= \frac{a}{4}\left(cot\frac{\pi}{n} + cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}\right) - \frac{a}{2}cot\frac{\pi}{n} = \frac{a}{4}cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - \frac{a}{4}cot\frac{\pi}{n}$$

$$= \frac{a}{4}\left(cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - cot\frac{\pi}{n}\right)$$

In right ΔMHN (figure 4)

$$MN = \sqrt{(MH)^2 + (NH)^2} = \sqrt{(EO)^2 + (NH)^2}$$

$$= \sqrt{\left(\frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}}\right)^2 + \left(\frac{a}{4}\left(cos\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} - cot\frac{\pi}{n}\right)\right)^2}$$

$$= a\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{8} + \frac{1}{16}\left(cos^2\frac{\pi}{n}\left(8 + cosec^2\frac{\pi}{n}\right) + cot^2\frac{\pi}{n} - 2cos\frac{\pi}{n}cot\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}\right)}$$

$$= \frac{a}{4}\sqrt{8 - 2cosec^2\frac{\pi}{n} + 2cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}} + 8cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n} + cot^2\frac{\pi}{n} - 2cos^2\frac{\pi}{n}cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}$$

$$= \frac{a}{4}\sqrt{8 - 2 - 2cot^2\frac{\pi}{n} + 8cos^2\frac{\pi}{n} + 2cot^2\frac{\pi}{n} + 2\left(1 - cos^2\frac{\pi}{n}\right)cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}$$

$$=\frac{a}{4}\sqrt{6+8cos^2\frac{\pi}{n}+2sin^2\frac{\pi}{n}cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}\quad=\frac{a}{4}\sqrt{6+8cos^2\frac{\pi}{n}+2\sqrt{8sin^2\frac{\pi}{n}+1}}$$

$$=\frac{a}{4}\sqrt{6+8cos^2\frac{\pi}{n}+2\sqrt{9-8cos^2\frac{\pi}{n}}}=\frac{a}{2}\sqrt{\frac{6+8cos^2\frac{\pi}{n}+2\sqrt{9-8cos^2\frac{\pi}{n}}}{4}}$$

$$\therefore\quad \boldsymbol{MN=\frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}}\qquad \dots\dots\dots\dots\dots.(III)$$

Now, area of ΔOMN can be calculated as follows (from figure 4)

$$area\ of\ \Delta OMN=\frac{1}{2}[(MN)\times(OG)]=\frac{1}{2}[(ON)\times(MH)]\ \Rightarrow\ (MN)\times(OG)=(ON)\times(MH)$$

$$\Rightarrow\ OG=\frac{(ON)\times(MH)}{MN}=\frac{\left(R_o cos\frac{\pi}{n}\right)\times\left(\frac{a}{2}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}\right)}{\left(\frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)}$$

$$=\frac{\frac{a}{4}\left(cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)cos\frac{\pi}{n}\sqrt{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}{\sqrt{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$=\frac{\frac{a}{4}cos\frac{\pi}{n}\sqrt{\left(4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}\right)\left(cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)^2}}{\sqrt{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$=\frac{\frac{a}{4}cos\frac{\pi}{n}\sqrt{32+16cosec^2\frac{\pi}{n}+16cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}{\sqrt{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$=acos\frac{\pi}{n}\sqrt{\frac{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}=a\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$$

$$\therefore\ OG = a\sqrt{\frac{2\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}+\cot\frac{\pi}{n}\sqrt{8\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}}}{3+4\cos^2\frac{\pi}{n}+\sqrt{9-8\cos^2\frac{\pi}{n}}}}\quad \ldots\ldots\ldots\ldots\ldots\ldots.(IV)$$

Normal distance (H_{n-gon}) of regular n-gonal faces from the centre of uniform polyhedron: The normal distance (H_{n-gon}) of each of 2 congruent regular n-gonal faces from the centre O of a uniform polyhedron is given as

$$H_{n-gon} = EO = \frac{a}{2}\sqrt{\frac{4-\mathrm{cosec}^2\frac{\pi}{n}+\mathrm{cosec}\frac{\pi}{n}\sqrt{8+\mathrm{cosec}^2\frac{\pi}{n}}}{2}}\qquad (from\ the\ eq(II)\ above)$$

$$\therefore\ H_{n-gon} = \frac{a}{2}\sqrt{\frac{4-\mathrm{cosec}^2\frac{\pi}{n}+\mathrm{cosec}\frac{\pi}{n}\sqrt{8+\mathrm{cosec}^2\frac{\pi}{n}}}{2}}\qquad \forall\ n\in N\ \&\ n\geq 3$$

It's clear that both the congruent regular n-gonal faces are at an equal normal distance H_{n-gon} from the centre of a uniform polyhedron.

Solid angle (ω_{n-gon}) subtended by each of 2 congruent regular n-gonal faces at the centre of uniform polyhedron: We know that the **solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane** is given by "**HCR's Theory of Polygon**" as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2+a^2\cot^2\frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle subtended by each regular n-gonal face at the centre of the uniform polyhedron as follows

$$\omega_{n-gon} = 2\pi - 2n\sin^{-1}\left(\frac{2(EO)\sin\frac{\pi}{n}}{\sqrt{4(EO)^2+a^2\cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{2\left(\frac{a}{2}\sqrt{\frac{4-\mathrm{cosec}^2\frac{\pi}{n}+\mathrm{cosec}\frac{\pi}{n}\sqrt{8+\mathrm{cosec}^2\frac{\pi}{n}}}{2}}\right)\sin\frac{\pi}{n}}{\sqrt{4\left(\frac{a}{2}\sqrt{\frac{4-\mathrm{cosec}^2\frac{\pi}{n}+\mathrm{cosec}\frac{\pi}{n}\sqrt{8+\mathrm{cosec}^2\frac{\pi}{n}}}{2}}\right)^2+a^2\cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sin\frac{\pi}{n}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}}{\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}+cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sin\frac{\pi}{n}\sqrt{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}{\sqrt{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}+2cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sqrt{4sin^2\frac{\pi}{n}+sin\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}-1}}{\sqrt{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}+2cosec^2\frac{\pi}{n}-2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4sin^2\frac{\pi}{n}-1+\sqrt{8sin^2\frac{\pi}{n}+1}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4-4cos^2\frac{\pi}{n}-1+\sqrt{8-8cos^2\frac{\pi}{n}+1}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)$$

$$\therefore\ \boldsymbol{\omega_{n-gon} = 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{3-4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)} \quad \forall\ \boldsymbol{n \in N\ \&\ n \geq 3}$$

$$\boldsymbol{Area\ of\ each\ n-gonal\ face,\ \ A_{n-gon} = \frac{1}{4}na^2cot\frac{\pi}{n}}$$

Normal distance (H_t) of trapezoidal faces from the centre of uniform polyhedron: The normal distance (H_t) of each of 2n congruent trapezoidal faces from the centre of uniform polyhedron is given as

$$H_t = OG = a\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}} \qquad (from\ the\ eq(IV)\ above)$$

$$\therefore\ \boldsymbol{H_t = a\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}} \quad \forall\ \boldsymbol{n \in N\ \&\ n \geq 3}$$

It's clear that all 2n congruent trapezoidal faces are at an equal normal distance H_t from the centre of any uniform polyhedron.

Solid angle (ω_t) subtended by each of 2n congruent trapezoidal faces at the centre of uniform polyhedron: Since a uniform polyhedron is a closed surface & we know that **the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi\ sr$ (Ste-radian)** hence the sum of solid angles subtended by **2 congruent regular n-gonal** & **2n congruent trapezoidal faces** at the centre of the uniform polyhedron must be $4\pi\ sr$. Thus we have

$$2[\omega_{n-gon}]+2n[\omega_{trapezium}]=4\pi\ \ or\ \ 2n[\omega_{trapezium}]=4\pi-2[\omega_{n-gon}]$$

$$\omega_{trapezium}=\frac{2\pi-\omega_{n-gon}}{n}=\frac{2\pi-\left[2\pi-2n\sin^{-1}\left(\sqrt{\frac{3-4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)\right]}{n}$$

$$=2\sin^{-1}\left(\sqrt{\frac{3-4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)$$

$$\therefore\ \ \boldsymbol{\omega_t=2\sin^{-1}\left(\sqrt{\frac{3-4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)}\qquad \forall\ \boldsymbol{n\in N\ \&\ n\geq 3}$$

Interior angles $(\alpha\ \&\ \beta)$ of the trapezoidal faces of uniform polyhedron: From above figures 1 & 2, let α be acute angle & β be obtuse angle. Acute angle α is determined as follows

$$\sin\angle BAD=\frac{MN}{AD}\Rightarrow sin\alpha=\frac{\frac{a}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}}{a}\qquad from\ eq(III)above$$

$$=\frac{1}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\quad or\ \ \alpha=\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)$$

$$\therefore\ \boldsymbol{Acute\ angle,\ \ \alpha=\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)}\qquad \forall\ \boldsymbol{n\in N\ \&\ n\geq 3}$$

In trapezoidal face ABCD, we know that the **sum of all interior angles (of a quadrilateral) is 360^o**

$$\therefore\ 2\alpha+2\beta=360^o\ \ or\ \ \beta=180^o-\alpha$$

$$\therefore\ \boldsymbol{Obtuse\ angle,\ \ \beta=180^o-\alpha}$$

Sides of the trapezoidal face of uniform polyhedron: All the sides of each trapezoidal face can be determined as follows (See figure 2 above)

$$AD = BC = CD = a \ \& \ AB = 2R_o \sin\frac{\pi}{n} = \frac{a}{2}\left(1 + \sqrt{1 + 8\sin^2\frac{\pi}{n}}\right) \quad (from\ eq(I)\ above)$$

$$Distance\ between\ parallel\ sides\ AB\ \&\ CD\ of\ trapezoidal\ face\ ABCD$$

$$\therefore\ MN = \frac{a}{2}\sqrt{\frac{3 + 4\cos^2\frac{\pi}{n} + \sqrt{9 - 8\cos^2\frac{\pi}{n}}}{2}} \qquad (from\ eq(III) above)$$

Hence, the **area of each of 2n congruent trapezoidal faces of a uniform polyhedron** is given as follows

$$Area\ of\ trapezium\ ABCD = \frac{1}{2}(sum\ of\ parallel\ sides) \times (normal\ distance\ between\ parallel\ sides)$$

$$\Rightarrow A_t = \frac{1}{2}(AB + CD)(MN) = \frac{1}{2}(R_o + a)\left(\frac{a}{2}\sqrt{\frac{3 + 4\cos^2\frac{\pi}{n} + \sqrt{9 - 8\cos^2\frac{\pi}{n}}}{2}}\right)$$

$$= \frac{a}{4}\left(\frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right) + a\right)\sqrt{\frac{3 + 4\cos^2\frac{\pi}{n} + \sqrt{9 - 8\cos^2\frac{\pi}{n}}}{2}}$$

$$= \frac{a^2}{16}\left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4\cos^2\frac{\pi}{n} + \sqrt{9 - 8\cos^2\frac{\pi}{n}}}{2}}$$

$$\therefore\ A_t = \frac{a^2}{16}\left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4\cos^2\frac{\pi}{n} + \sqrt{9 - 8\cos^2\frac{\pi}{n}}}{2}}$$

Important parameters of a uniform polyhedron:

1. **Inner (inscribed) radius (R_i):** It is the radius of the largest sphere inscribed (trapped inside) by a uniform polyhedron. The largest inscribed sphere either touches both the congruent regular n-gonal faces or touches all 2n congruent trapezoidal faces depending on the value of no. of sides n of the regular polygonal face & is equal to the minimum value out of H_{n-gon} & H_t & is given as follows

$$R_i = Min(H_{n-gon},\ H_t)$$

$$Where, \qquad H_{n-gon} = \frac{a}{2}\sqrt{\frac{4 - cosec^2\frac{\pi}{n} + cosec\frac{\pi}{n}\sqrt{8 + cosec^2\frac{\pi}{n}}}{2}} \qquad \&$$

$$H_t = a\sqrt{\frac{2cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n} + cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n} + cot^2\frac{\pi}{n}}}{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}}$$

2. **Outer (circumscribed) radius (R_o):** It is the radius of the smallest sphere circumscribing a uniform polyhedron or it's the radius of a spherical surface passing through all 3n vertices of a uniform polyhedron. It is given from eq(I) as follows

$$\boldsymbol{R_o = \frac{a}{4}\left(cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)}$$

3. **Surface area (A_s):** We know that a uniform polyhedron has 2 congruent regular n-gonal faces & 2n congruent trapezoidal faces. Hence, its surface area is given as follows

$$A_s = 2(area\ of\ regular\ polygon) + 2n(area\ of\ trapezium\ ABCD) \qquad (see\ figure\ 2\ above)$$

We know that **area of any regular n-polygon** with each side of length a is given as

$$\boldsymbol{A = \frac{1}{4}na^2cot\frac{\pi}{n}}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 2 \times \left(\frac{1}{4}na^2cot\frac{\pi}{n}\right) + 2n \times \left(\frac{1}{2}(AB + CD)(MN)\right)$$

$$= 2 \times \left(\frac{1}{4}na^2cot\frac{\pi}{n}\right) + 2n \times \left(\frac{a^2}{16}\left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2}}\right)$$

$$= \frac{na^2}{8}\left(4cot\frac{\pi}{n} + \left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2}}\right)$$

$$\boldsymbol{\therefore\ A_s = \frac{na^2}{8}\left(4cot\frac{\pi}{n} + \left(4 + cosec\frac{\pi}{n} + \sqrt{8 + cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3 + 4cos^2\frac{\pi}{n} + \sqrt{9 - 8cos^2\frac{\pi}{n}}}{2}}\right)}$$

4. **Volume (V):** We know that a uniform polyhedron has 2 congruent regular n-gonal & 2n congruent trapezoidal faces. Hence, the **volume (V) of the uniform polyhedron** is the **sum of volumes of all its $(2n + 2)$ elementary right pyramids** with regular n-gonal & trapezoidal bases (faces) given as follows

$$V = 2(volume\ of\ right\ pyramid\ with\ regular\ polygonal\ base) + 2n(volume\ of\ right\ pyramid\ with\ trapezoidal\ base\ ABCD)$$

$$= 2\left(\frac{1}{3}(area\ of\ regular\ polygon) \times H_{n-gon}\right) + 2n\left(\frac{1}{3}(area\ of\ trapezium\ ABCD) \times H_t\right)$$

$$= 2\left(\frac{1}{3}\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)\times\frac{a}{2}\sqrt{\frac{4-\operatorname{cosec}^2\frac{\pi}{n}+\operatorname{cosec}\frac{\pi}{n}\sqrt{8+\operatorname{cosec}^2\frac{\pi}{n}}}{2}}\right)$$

$$+2n\left(\frac{1}{3}\left(\frac{a^2}{16}\left(4+\operatorname{cosec}\frac{\pi}{n}+\sqrt{8+\operatorname{cosec}^2\frac{\pi}{n}}\right)\sqrt{\frac{3+4\cos^2\frac{\pi}{n}+\sqrt{9-8\cos^2\frac{\pi}{n}}}{2}}\right)\right.$$

$$\left.\times a\sqrt{\frac{2\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}+\cot\frac{\pi}{n}\sqrt{8\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}}}{3+4\cos^2\frac{\pi}{n}+\sqrt{9-8\cos^2\frac{\pi}{n}}}}\right)$$

$$=\frac{1}{12}na^3\cot\frac{\pi}{n}\sqrt{\frac{4-\operatorname{cosec}^2\frac{\pi}{n}+\operatorname{cosec}\frac{\pi}{n}\sqrt{8+\operatorname{cosec}^2\frac{\pi}{n}}}{2}}$$

$$+\frac{1}{24}na^3\left(4+\operatorname{cosec}\frac{\pi}{n}+\sqrt{8+\operatorname{cosec}^2\frac{\pi}{n}}\right)\sqrt{\frac{2\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}+\cot\frac{\pi}{n}\sqrt{8\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}}}{2}}$$

$$\therefore\ V=\frac{1}{24}na^3\left(2\cot\frac{\pi}{n}\sqrt{\frac{4-\operatorname{cosec}^2\frac{\pi}{n}+\operatorname{cosec}\frac{\pi}{n}\sqrt{8+\operatorname{cosec}^2\frac{\pi}{n}}}{2}}\right.$$

$$\left.+\left(4+\operatorname{cosec}\frac{\pi}{n}+\sqrt{8+\operatorname{cosec}^2\frac{\pi}{n}}\right)\sqrt{\frac{2\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}+\cot\frac{\pi}{n}\sqrt{8\cos^2\frac{\pi}{n}+\cot^2\frac{\pi}{n}}}{2}}\right)$$

$\forall\ \boldsymbol{n}\in\boldsymbol{N}\ \&\ \boldsymbol{n}\geq \mathbf{3}$

5. Mean radius ($\boldsymbol{R_m}$): It is the radius of the sphere having a volume equal to that of a uniform polyhedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ the\ uniform\ polyhedron$$

$$\frac{4}{3}\pi(R_m)^3 = V \quad\Rightarrow\quad (R_m)^3=\frac{3V}{4\pi}\quad\Rightarrow\quad \boldsymbol{R_m}=\left(\frac{\mathbf{3V}}{\mathbf{4\pi}}\right)^{\frac{1}{3}}$$

For finite value of edge length a of regular n-gonal face $\Rightarrow$ $\boldsymbol{R_i < R_m < R_o}$

Hence, by setting different values of no. of sides $n = 3, 4, 5, 6, 7$ … … … … … .. we can find out all the important parameters of any regular polyhedral with known value of side a of regular n-gonal face.

Conclusions: All the formula above are generalised which are applicable to calculate the important parameters, of any uniform polyhedron having 2 congruent regular n-gonal faces, 2n congruent trapezoidal faces with three equal sides, 5n edges & 3n vertices lying on a spherical surface, such as solid angle subtended by each face at the centre, normal distance of each face from the centre, inner radius, outer radius, mean radius, surface area & volume.

*Let there be any **uniform polyhedron** having $\mathbf{2}$ congruent **regular n-gonal faces** each with edge length a, $\mathbf{2n}$ congruent **trapezoidal faces** each with three sides equal to a & forth equal to $2R_o \sin {}^{\pi}/_{n}$, $\mathbf{5n}$ **edges** and $\mathbf{3n}$ **vertices lying on a spherical surface** then all its important parameters are calculated as tabulated below*

Congruent polygonal faces	No. of faces	Normal distance of each face from the centre of the uniform polyhedron	Solid angle subtended by each face at the centre of the uniform polyhedron (in sr)
Regular polygon	2	$\frac{a}{2}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}$	$2\pi-2n\sin^{-1}\sqrt{\frac{3-4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}$
Trapezium	$2n$	$a\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}}$	$2\sin^{-1}\left(\sqrt{\frac{3-4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2+cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}}\right)$
Inner (inscribed) radius (R_i)		$R_i = Minimum\ normal\ distance\ of\ any\ face\ from\ the\ centre$	
Outer (circumscribed) radius (R_o)		$R_o=\frac{a}{4}\left(cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)$	
Mean radius (R_m)		$R_m=\left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$	
Surface area (A_s)		$A_s=\frac{na^2}{8}\left(4cot\frac{\pi}{n}+\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{3+4cos^2\frac{\pi}{n}+\sqrt{9-8cos^2\frac{\pi}{n}}}{2}}\right)$	
Volume (V)		$V=\frac{1}{24}na^3\left(2cot\frac{\pi}{n}\sqrt{\frac{4-cosec^2\frac{\pi}{n}+cosec\frac{\pi}{n}\sqrt{8+cosec^2\frac{\pi}{n}}}{2}}+\left(4+cosec\frac{\pi}{n}+\sqrt{8+cosec^2\frac{\pi}{n}}\right)\sqrt{\frac{2cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}+cot\frac{\pi}{n}\sqrt{8cos^2\frac{\pi}{n}+cot^2\frac{\pi}{n}}}{2}}\right)$	

Mathematical analysis of uniform polyhedra with 2 regular n-gonal & 2n trapezoidal faces
(Generalized formula for uniform polyhedra with regular polygonal & trapezoidal faces)

Note: Above articles had been developed *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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