$\boldsymbol{Mathematical\ Analysis\ of\ Great\ Rhombicuboctahedron}$

$\boldsymbol{Application\ of\ HCR's\ Theory\ of\ Polygon}$

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Introduction: A great rhombicuboctahedron is an Archimedean solid which has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each having equal edge length. It has 72 edges & 48 vertices lying on a spherical surface with a certain radius. It is created/generated by expanding a truncated cube having 8 equilateral triangular faces & 6 regular octagonal faces. Thus by the expansion, each of 12 originally truncated edges changes into a square face, each of 8 triangular faces of the original solid changes into a regular hexagonal face & 6 regular octagonal faces of original solid remain unchanged i.e. octagonal faces are shifted radially. Thus a solid with 12 squares, 8 hexagonal & 6 octagonal faces, is obtained which is called **great rhombicuboctahedron** which is an **Archimedean solid**. (See figure 1), thus we have

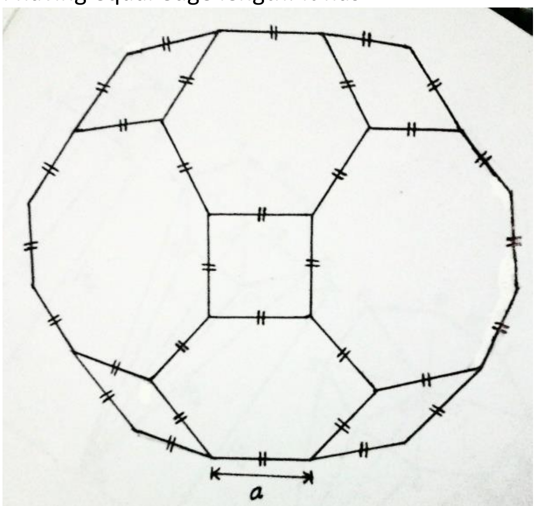

Figure 1: A great rhombicuboctahedron having 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of equal edge length $\boldsymbol{a}$

$no. of\ square\ faces = 12,\ no. of\ regular\ hexagonal\ faces = 8$

$no. of\ regular\ octagonal\ faces = 6,$

$no. of\ edges =$
$6(no. of\ edges\ in\ an\ octagonal\ face) + 2(no. of\ square\ faces) =$
$6(8) + 2(12) = 48 + 24 = 72$

$no. of\ vertices = 6(no. of\ vertices\ in\ an\ octagonal\ face) = 6(8) = 48$

We would apply HCR's Theory of Polygon to derive a mathematical relationship between radius R of the spherical surface passing through all 48 vertices & the edge length a of a great rhombicuboctahedron for calculating its important parameters such as normal distance of each face, surface area, volume etc.

Derivation of outer (circumscribed) radius $(\boldsymbol{R_o})$ of great rhombicuboctahedron:

Let R_o be the radius of the spherical surface passing through all 48 vertices of a great rhombicuboctahedron with edge length a & the centre O. Consider a square face ABCD with centre O_1, regular hexagonal face EFGHIJ with centre O_2 & regular octagonal face KLMNPQRS with centre O_3 (see the figure 2 below)

Normal distance $(\boldsymbol{H_s})$ of square face ABCD from the centre O of the great rhombicuboctahedron is calculated as follows

In right ΔAO_1O (figure 2)

$$OO_1 = \sqrt{(OA)^2 - (O_1A)^2} \qquad \left(\boldsymbol{O_1A = \frac{a}{\sqrt{2}}} = circumscribed\ radius\ of\ square\right)$$

$$\Rightarrow \boldsymbol{H_s} = \sqrt{(R_o)^2 - \left(\frac{a}{\sqrt{2}}\right)^2} = \sqrt{\frac{\boldsymbol{2R_o^2 - a^2}}{\boldsymbol{2}}} \qquad \dots\dots\dots\dots.(I)$$

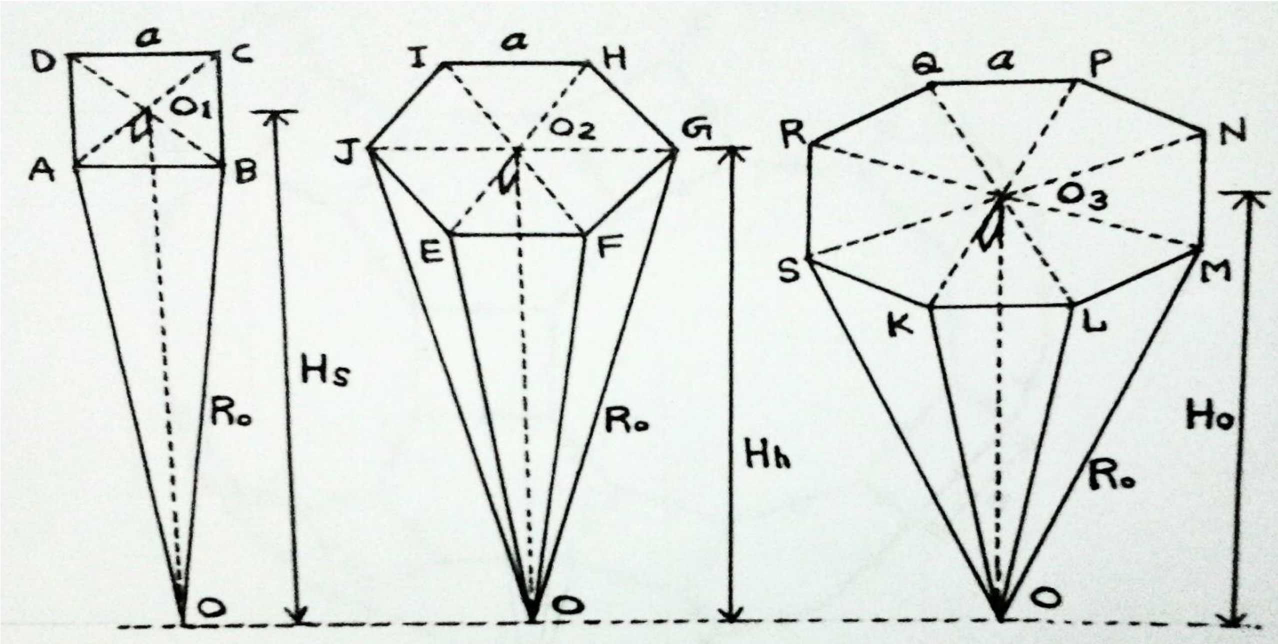

Figure 2: A square face ABCD, regular hexagonal face EFGHIJ & regular octagonal face KLMNPQRS each of edge length a

We know that the solid angle (ω) subtended by any regular polygon with each side of length a at any point lying at a distance H on the vertical axis passing through the centre of plane is given by "**HCR's Theory of Polygon**" as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get the solid angle $(\boldsymbol{\omega_s})$ subtended by each square face (ABCD) at the centre of great rhombicuboctahedron as follows

$$\omega_s = 2\pi - 2\times 4\sin^{-1}\left(\frac{2\left(\sqrt{\frac{2{R_o}^2 - a^2}{2}}\right)\sin\frac{\pi}{4}}{\sqrt{4\left(\sqrt{\frac{2{R_o}^2 - a^2}{2}}\right)^2 + a^2\cot^2\frac{\pi}{4}}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2}\sqrt{2{R_o}^2 - a^2}\times\frac{1}{\sqrt{2}}}{\sqrt{4{R_o}^2 - 2a^2 + a^2(1)^2}}\right) = 2\pi - 8\sin^{-1}\left(\frac{\sqrt{2{R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2{R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right)$$

$$\boldsymbol{Let\ \frac{R_o}{a} = x > 1\ (any\ arbitrary\ variable)}$$

$$\Rightarrow \boldsymbol{\omega_s} = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{R_o}{a}\right)^2 - 1}{4\left(\frac{R_o}{a}\right)^2 - 1}}\right) = \boldsymbol{2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2 - 1}{4x^2 - 1}}\right)} \quad \ldots\ldots\ldots\ldots(II)$$

Similarly, **Normal distance** $(\boldsymbol{H_h})$ of **regular hexagonal face** EFGHIJ from the centre O of great rhombicuboctahedron is calculated as follows

In right ΔEO_2O (figure 2)

$$OO_2 = \sqrt{(OE)^2 - (O_2E)^2} \qquad (\boldsymbol{O_2E} = \boldsymbol{a} = circumscribed\ radius\ of\ regular\ hexagon)$$

$$\Rightarrow \boldsymbol{H_h} = \sqrt{(R_o)^2 - (a)^2} = \sqrt{\boldsymbol{R_o}^2 - \boldsymbol{a}^2} \qquad \dots\dots\dots\dots\dots (III)$$

Hence, by substituting all the corresponding values, the solid angle $(\boldsymbol{\omega_h})$ subtended by each regular hexagonal face (EFGHIJ) at the centre of great rhombicuboctahedron is given as follows

$$\omega_h = 2\pi - 2 \times 6 \sin^{-1}\left(\frac{2\left(\sqrt{{R_o}^2 - a^2}\right) sin\frac{\pi}{6}}{\sqrt{4\left(\sqrt{{R_o}^2 - a^2}\right)^2 + a^2cot^2\frac{\pi}{6}}}\right)$$

$$= 2\pi - 12 \sin^{-1}\left(\frac{2\sqrt{{R_o}^2 - a^2} \times \frac{1}{2}}{\sqrt{4{R_o}^2 - 4a^2 + a^2\left(\sqrt{3}\right)^2}}\right) = 2\pi - 12 \sin^{-1}\left(\frac{\sqrt{{R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right)$$

$$\Rightarrow \boldsymbol{\omega_h} = 2\pi - 12 \sin^{-1}\left(\sqrt{\frac{{R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right) = \boldsymbol{2\pi - 12\sin^{-1}}\left(\sqrt{\frac{\boldsymbol{x^2 - 1}}{\boldsymbol{4x^2 - 1}}}\right) \qquad \dots\dots\dots\dots\dots (IV)$$

Similarly, **Normal distance** $(\boldsymbol{H_o})$ of **regular octagonal face** KLMNPQRS from the centre O of great rhombicuboctahedron is calculated as follows

In right ΔKO_3O (figure 2)

$$\Rightarrow \ OO_3 = \sqrt{(OE)^2 - (O_3K)^2}$$

$$\boldsymbol{O_3K} = circumscribed\ radius\ of\ regular\ octagon = \frac{\boldsymbol{a}}{\boldsymbol{2}}\boldsymbol{cosec22.5^o} = \frac{\boldsymbol{a}}{\boldsymbol{2}}\sqrt{\boldsymbol{4 + 2\sqrt{2}}}$$

$$\Rightarrow \boldsymbol{H_o} = \sqrt{(R_o)^2 - \left(\frac{a}{2}\sqrt{4 + 2\sqrt{2}}\right)^2} = \sqrt{\frac{\boldsymbol{4R_o}^2 - \left(\boldsymbol{4 + 2\sqrt{2}}\right)\boldsymbol{a^2}}{\boldsymbol{4}}} \qquad \dots\dots\dots\dots\dots (V)$$

Hence, by substituting all the corresponding values, the solid angle $(\boldsymbol{\omega_o})$ subtended by each regular octagonal face (KLMNPQRS) at the centre of great rhombicuboctahedron is given as follows

$$\omega_o = 2\pi - 2 \times 8 \sin^{-1}\left(\frac{2\left(\sqrt{\frac{4{R_o}^2 - \left(4 + 2\sqrt{2}\right)a^2}{4}}\right) sin\frac{\pi}{8}}{\sqrt{4\left(\sqrt{\frac{4{R_o}^2 - \left(4 + 2\sqrt{2}\right)a^2}{4}}\right)^2 + a^2cot^2\frac{\pi}{8}}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\frac{\sqrt{4{R_o}^2 - (4+2\sqrt{2})a^2}\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{\sqrt{4{R_o}^2 - (4+2\sqrt{2})a^2 + a^2(\sqrt{2}+1)^2}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\frac{\sqrt{4\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right){R_o}^2 - (4+2\sqrt{2})\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)a^2}}{\sqrt{4{R_o}^2 - (4+2\sqrt{2})a^2 + (3+2\sqrt{2})a^2}}\right) = 2\pi - 16\sin^{-1}\left(\frac{\sqrt{(2-\sqrt{2}){R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right)$$

$$\Rightarrow \boldsymbol{\omega_o} = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2}){R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right) = \mathbf{2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2 - 1}{4x^2 - 1}}\right)} \quad \ldots\ldots\ldots\ldots\ldots (VI)$$

Since a great rhombicuboctahedron is a **closed surface** & we know that **the total solid angle, subtended by any closed surface at any point lying inside it, is $4\pi\ sr$ (Ste-radian)** hence the sum of solid angles subtended by **12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces** at the centre of the great rhombicuboctahedron must be $4\pi\ sr$. Thus we have

$$\mathbf{12[\omega_s] + 8[\omega_h] + 6[\omega_o] = 4\pi}$$

Now, by substituting the values of $\boldsymbol{\omega_s}$, $\boldsymbol{\omega_h}$ & $\boldsymbol{\omega_o}$ from eq(II), (IV) & (VI) in the above expression we get

$$12\left[2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)\right] + 8\left[2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)\right] + 6\left[2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)\right] = 4\pi$$

$$\Rightarrow 96\left[\sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)\right] = 52\pi - 4\pi = 48\pi$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right) = \frac{48\pi}{96} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right) + \sin^{-1}\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right) = \frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{1-\left(\sqrt{\frac{x^2-1}{4x^2-1}}\right)^2} + \sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{1-\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\right)^2}\right) = \cos^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)$$

$$\left(\boldsymbol{since},\ \ \mathbf{\sin^{-1}X + \sin^{-1}Y = \sin^{-1}\left(X\sqrt{1-Y^2} + Y\sqrt{1-X^2}\right)} \quad \& \quad \mathbf{\frac{\pi}{2} - \sin^{-1}X = \cos^{-1}X}\right)$$

$$\Rightarrow \sin^{-1}\left(\sqrt{\frac{2x^2-1}{4x^2-1}}\sqrt{\frac{3x^2}{4x^2-1}}+\sqrt{\frac{x^2-1}{4x^2-1}}\sqrt{\frac{2x^2}{4x^2-1}}\right)=\sin^{-1}\left(\sqrt{1-\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right)^2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}\right)=\sin^{-1}\left(\sqrt{\frac{(2+\sqrt{2})x^2}{4x^2-1}}\right)=\sin^{-1}\left(x\sqrt{\frac{(2+\sqrt{2})}{4x^2-1}}\right)$$

$$\Rightarrow \frac{x\sqrt{3(2x^2-1)}}{4x^2-1}+\frac{x\sqrt{2(x^2-1)}}{4x^2-1}=x\sqrt{\frac{(2+\sqrt{2})}{4x^2-1}}$$

$$\Rightarrow \sqrt{3(2x^2-1)}+\sqrt{2(x^2-1)}=\sqrt{(2+\sqrt{2})(4x^2-1)}$$

$$\Rightarrow \left(\sqrt{3(2x^2-1)}+\sqrt{2(x^2-1)}\right)^2=\left(\sqrt{(2+\sqrt{2})(4x^2-1)}\right)^2$$

$$\Rightarrow 3(2x^2-1)+2(x^2-1)+2\sqrt{6(x^2-1)(2x^2-1)}=(2+\sqrt{2})(4x^2-1)$$

$$\Rightarrow 4(2+\sqrt{2})x^2-(2+\sqrt{2})-8x^2+5=2\sqrt{6(x^2-1)(2x^2-1)}$$

$$\Rightarrow \left(4\sqrt{2}x^2+(3-\sqrt{2})\right)^2=\left(2\sqrt{6(x^2-1)(2x^2-1)}\right)^2$$

$$\Rightarrow 32x^4+8\sqrt{2}(3-\sqrt{2})x^2+(11-6\sqrt{2})=24(x^2-1)(2x^2-1)=48x^4-72x^2+24$$

$$\Rightarrow 16x^4-8(7+3\sqrt{2})x^2+(13+6\sqrt{2})=0$$

Now, solving the above **biquadratic equation for the values of $x > 1$** as follows

$$\Rightarrow x^2=\frac{-\left(-8(7+3\sqrt{2})\right)\pm\sqrt{\left(-8(7+3\sqrt{2})\right)^2-4(16)(13+6\sqrt{2})}}{2\times 16}$$

$$=\frac{8(7+3\sqrt{2})\pm 8\sqrt{67+42\sqrt{2}-(13+6\sqrt{2})}}{32}=\frac{(7+3\sqrt{2})\pm\sqrt{54+36\sqrt{2}}}{4}=\frac{(7+3\sqrt{2})\pm 3\sqrt{6+4\sqrt{2}}}{4}$$

$$=\frac{(7+3\sqrt{2})\pm 3\sqrt{(2+\sqrt{2})^2}}{4}=\frac{(7+3\sqrt{2})\pm 3(2+\sqrt{2})}{4}$$

1. **Taking positive sign, we have**

$$x^2=\frac{(7+3\sqrt{2})+3(2+\sqrt{2})}{4}=\frac{13+6\sqrt{2}}{4}\quad or\ \ x=\sqrt{\frac{13+6\sqrt{2}}{4}}=\frac{\mathbf{1}}{\mathbf{2}}\sqrt{\mathbf{13+6\sqrt{2}}}$$

Since, $x > 1$ hence, the above value is acceptable.

2. **Taking negative sign, we have**

$$x^2 = \frac{(7+3\sqrt{2})-3(2+\sqrt{2})}{4} = \frac{1}{4} \quad or \quad x = \sqrt{\frac{1}{4}} = \frac{1}{2} \quad \Rightarrow \boldsymbol{x < 1 \;\; but\; x > 1\; (required\; condition)}$$

Hence, the above value is discarded, now we have

$$x = \frac{1}{2}\sqrt{13+6\sqrt{2}} \;\Rightarrow\; \frac{R_o}{a} = x = \frac{1}{2}\sqrt{13+6\sqrt{2}} \quad or\; R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}}$$

Hence, the **outer (circumscribed) radius (R_o) of a great rhombicuboctahedron with edge length a** is given as

$$\boldsymbol{R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}} \approx 2.317610913a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots. \; (VII)$$

Normal distance (H_s) of square faces from the centre of great rhombicuboctahedron: The normal distance (H_s) of each of 12 congruent square faces from the centre O of a great rhombicuboctahedron is given from eq(I) as follows

$$H_s = OO_1 = \sqrt{\frac{2{R_o}^2 - a^2}{2}} = \sqrt{\frac{2\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}{2}} = a\sqrt{\frac{13+6\sqrt{2}-2}{4}} = \frac{a}{2}\sqrt{11+6\sqrt{2}} = \frac{a}{2}\sqrt{\left(3+\sqrt{2}\right)^2}$$
$$= \frac{(3+\sqrt{2})a}{2}$$

$$\therefore \; \boldsymbol{H_s = \frac{(3+\sqrt{2})a}{2} \approx 2.207106781a}$$

It's clear that all 12 congruent square faces are at an equal normal distance H_s from the centre of any great rhombicuboctahedron.

Solid angle (ω_s) subtended by each of the square faces at the centre of great rhombicuboctahedron: solid angle $(\boldsymbol{\omega_s})$ subtended by each square face is given from eq(II) as follows

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2x^2-a^2}{4x^2-a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2{R_o}^2-a^2}{4{R_o}^2-a^2}}\right) \qquad \left(since, \;\; \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_s = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{2\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{13+6\sqrt{2}-2}{2(13+6\sqrt{2}-1)}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{11+6\sqrt{2}}{12(2+\sqrt{2})}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{(11+6\sqrt{2})(2-\sqrt{2})}{12(2+\sqrt{2})(2-\sqrt{2})}}\right) = 2\pi - 8\sin^{-1}\left(\sqrt{\frac{10+\sqrt{2}}{24}}\right)$$

$$\therefore \; \boldsymbol{\omega_s = 2\pi - 8\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{10+\sqrt{2}}{6}}\right) = 4\sin^{-1}\left(\frac{2-\sqrt{2}}{12}\right) \approx 0.195339779\; sr}$$

Normal distance (H_h) of regular hexagonal faces from the centre of great rhombicuboctahedron: The normal distance (H_h) of each of 8 congruent regular hexagonal faces from the centre O of a great rhombicuboctahedron is given from eq(III) as follows

$$H_h = OO_2 = \sqrt{R_o{}^2 - a^2} = \sqrt{\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2} = a\sqrt{\frac{13+6\sqrt{2}-4}{4}} = \frac{a}{2}\sqrt{9+6\sqrt{2}} = \frac{a}{2}\sqrt{3\left(1+\sqrt{2}\right)^2}$$
$$= \frac{\sqrt{3}\left(1+\sqrt{2}\right)a}{2}$$

$$\therefore\ \boldsymbol{H_h = \frac{\sqrt{3}\left(1+\sqrt{2}\right)a}{2} \approx 2.090770275a}$$

It's clear that all 8 congruent regular hexagonal faces are at an equal normal distance H_h from the centre of any great rhombicuboctahedron.

Solid angle (ω_h) subtended by each of the regular hexagonal faces at the centre of great rhombicuboctahedron: solid angle $(\boldsymbol{\omega_h})$ subtended by each regular hexagonal face is given from eq(IV) as follows

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{x^2-a^2}{4x^2-a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{R_o{}^2-a^2}{4R_o{}^2-a^2}}\right) \qquad \left(since,\ \boldsymbol{x = \frac{R_o}{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_h = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - a^2}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{13+6\sqrt{2}-4}{4\left(13+6\sqrt{2}-1\right)}}\right)$$

$$= 2\pi - 12\sin^{-1}\left(\sqrt{\frac{3+2\sqrt{2}}{8\left(2+\sqrt{2}\right)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{\left(3+2\sqrt{2}\right)\left(2-\sqrt{2}\right)}{8\left(2+\sqrt{2}\right)\left(2-\sqrt{2}\right)}}\right) = 2\pi - 12\sin^{-1}\left(\sqrt{\frac{2+\sqrt{2}}{16}}\right)$$

$$\therefore\ \boldsymbol{\omega_h = 2\pi - 12\sin^{-1}\left(\frac{\sqrt{2+\sqrt{2}}}{4}\right) \approx 0.5210126\ sr}$$

Normal distance (H_o) of regular octagonal faces from the centre of great rhombicuboctahedron: The normal distance (H_o) of each of 6 congruent regular octagonal faces from the centre O of a great rhombicuboctahedron is given from eq(V) as follows

$$H_o = OO_3 = \sqrt{\frac{4R_o{}^2 - \left(4+2\sqrt{2}\right)a^2}{4}} = \sqrt{\frac{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2 - \left(4+2\sqrt{2}\right)a^2}{4}} = \frac{a}{2}\sqrt{13+6\sqrt{2}-\left(4+2\sqrt{2}\right)}$$
$$= \frac{a}{2}\sqrt{9+4\sqrt{2}} = \frac{a}{2}\sqrt{\left(1+2\sqrt{2}\right)^2} = \frac{\left(1+2\sqrt{2}\right)a}{2}$$

$$\therefore\ \boldsymbol{H_o = \frac{\left(1+2\sqrt{2}\right)a}{2} \approx 1.914213562a}$$

It's clear that all 6 congruent regular octagonal faces are at an equal normal distance H_o from the centre of any great rhombicuboctahedron.

Solid angle (ω_o) subtended by each of the regular octagonal faces at the centre of great rhombicuboctahedron: solid angle $(\boldsymbol{\omega_o})$ subtended by each regular octagonal face is given from eq(VI) as follows

$$\omega_o = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})x^2-1}{4x^2-1}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2}){R_o}^2-a^2}{4{R_o}^2-a^2}}\right) \quad \left(since,\ \boldsymbol{x}=\frac{\boldsymbol{R_o}}{\boldsymbol{a}}\right)$$

Hence, by substituting the corresponding value of R_o in the above expression, we get

$$\omega_o = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2-a^2}{4\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^2-a^2}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(2-\sqrt{2})(13+6\sqrt{2})-4}{4(13+6\sqrt{2}-1)}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\sqrt{\frac{10-\sqrt{2}}{24(2+\sqrt{2})}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(10-\sqrt{2})(2-\sqrt{2})}{24(2+\sqrt{2})(2-\sqrt{2})}}\right)$$

$$= 2\pi - 16\sin^{-1}\left(\sqrt{\frac{22-12\sqrt{2}}{48}}\right) = 2\pi - 16\sin^{-1}\left(\sqrt{\frac{(3-\sqrt{2})^2}{24}}\right) = 2\pi - 16\sin^{-1}\left(\frac{3-\sqrt{2}}{2\sqrt{6}}\right)$$

$$\therefore\ \boldsymbol{\omega_o = 2\pi - 16\sin^{-1}\left(\frac{3-\sqrt{2}}{2\sqrt{6}}\right) \approx 1.009032076\ sr}$$

It's clear from the above results that the solid angle subtended by each of 6 regular octagonal faces is greater than the solid angle subtended by each of 12 square faces & each of 8 regular hexagonal faces at the centre of any great rhombicuboctahedron.

It's also clear from the above results that $\boldsymbol{H_s > H_h > H_o}$ i.e. the normal distance (H_s) of square faces is greater than the normal distance H_h of the regular hexagonal faces & the normal distance H_o of the regular octagonal faces from the centre of a great rhombicuboctahedron i.e. **regular octagonal faces are closer to the centre as compared to the square & regular hexagonal faces in any great rhombicuboctahedron**.

Important parameters of a great rhombicuboctahedron:

1. **Inner (inscribed) radius $(\boldsymbol{R_i})$:** It is the radius of the largest sphere inscribed (trapped inside) by a great rhombicuboctahedron. The largest inscribed sphere always touches all 6 congruent regular octagonal faces but does not touch any of 12 congruent square & any of 8 congruent regular hexagonal faces at all since all 6 octagonal faces are closest to the centre in all the faces. Thus, inner radius is always equal to the normal distance (H_o) of regular octagonal faces from the centre of a great rhombicuboctahedron & is given as follows

$$\boldsymbol{R_i = H_o = \frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a}$$

Hence, the **volume of inscribed sphere** is given as

$$V_{inscribed} = \frac{4}{3}\pi(R_i)^3 = \frac{4}{3}\pi\left(\frac{(1+2\sqrt{2})a}{2}\right)^3 \approx 29.38054016a^3$$

2. **Outer (circumscribed) radius (R_o):** It is the radius of the smallest sphere circumscribing a great rhombicuboctahedron or it's the radius of a spherical surface passing through all 48 vertices of a great rhombicuboctahedron. It is from the eq(VII) as follows

$$R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}} \approx 2.317610913a$$

Hence, the **volume of circumscribed sphere** is given as

$$V_{circumscribed} = \frac{4}{3}\pi(R_o)^3 = \frac{4}{3}\pi\left(\frac{a}{2}\sqrt{13+6\sqrt{2}}\right)^3 = 52.14470211a^3$$

3. **Surface area (A_s):** We know that a great rhombicuboctahedron has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length a. Hence, its surface area is given as follows

$$A_s = 12(area\ of\ square) + 8(area\ of\ regular\ hexagon) + 6(area\ of\ regular\ octagon)$$

We know that **area of any regular n-polygon** with each side of length a is given as

$$A = \frac{1}{4}na^2\cot\frac{\pi}{n}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 12\times\left(\frac{1}{4}\times 4a^2\cot\frac{\pi}{4}\right) + 8\times\left(\frac{1}{4}\times 6a^2\cot\frac{\pi}{6}\right) + 6\times\left(\frac{1}{4}\times 8a^2\cot\frac{\pi}{8}\right)$$

$$= 12a^2 + 12\sqrt{3}a^2 + 12(1+\sqrt{2})a^2 = 12(2+\sqrt{2}+\sqrt{3})a^2$$

$$A_s = 12(2+\sqrt{2}+\sqrt{3})a^2 \approx 61.75517244a^2$$

4. **Volume (V):** We know that a great rhombicuboctahedron with edge length a has 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces. Hence, the volume (V) of the great rhombicuboctahedron is the sum of volumes of all its elementary right pyramids with square base, regular hexagonal base & regular octagonal base (face) (see figure 2 above) & is given as follows

$$V = 12(volume\ of\ right\ pyramid\ with\ square\ base) + 8(volume\ of\ right\ pyramid\ with\ regular\ hexagonal\ base) + 6(volume\ of\ right\ pyramid\ with\ regular\ octagonal\ base)$$

$$= 12\left(\frac{1}{3}(area\ of\ square)\times H_s\right) + 8\left(\frac{1}{3}(area\ of\ regular\ hexagon)\times H_h\right) + 6\left(\frac{1}{3}(area\ of\ regular\ octagon)\times H_o\right)$$

$$= 12\left(\frac{1}{3}\left(\frac{1}{4}\times 4a^2\cot\frac{\pi}{4}\right)\times\frac{(3+\sqrt{2})a}{2}\right) + 8\left(\frac{1}{3}\left(\frac{1}{4}\times 6a^2\cot\frac{\pi}{6}\right)\times\frac{\sqrt{3}(1+\sqrt{2})a}{2}\right) + 6\left(\frac{1}{3}\left(\frac{1}{4}\times 8a^2\cot\frac{\pi}{8}\right)\times\frac{(1+2\sqrt{2})a}{2}\right)$$

$$= 2(3+\sqrt{2})a^3 + 6(1+\sqrt{2})a^3 + 2(1+\sqrt{2})(1+2\sqrt{2})a^3 = (22+14\sqrt{2})a^3$$

$$\boldsymbol{V = (22+14\sqrt{2})a^3 \approx 41.79898987a^3}$$

5. **Mean radius (R_m):** It is the radius of the sphere having a volume equal to that of a great rhombicuboctahedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ the\ great\ rhombicuboctahedron$$

$$\frac{4}{3}\pi(R_m)^3 = (22+14\sqrt{2})a^3 \ \Rightarrow\ (R_m)^3 = \frac{3(11+7\sqrt{2})a^3}{2\pi} \quad or\ \ R_m = a\left(\frac{3(11+7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}}$$

$$\boldsymbol{R_m = a\left(\frac{3(11+7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}} \approx 2.15290926a}$$

It's clear from above results that $\boldsymbol{R_i < R_m < R_o}$

6. **Dihedral angles between the adjacent faces:** In order to calculate dihedral angles between the different adjacent faces with a common edge in a great rhombicuboctahedron, let's consider one-by-one all three pairs of adjacent faces with a common edge as follows

a. **Angle between square face & regular hexagonal face:** Draw the perpendiculars OO_1 & OO_2 from the centre O of great rhombicuboctahedron to the square face & the regular hexagonal face which have a common edge (See figure 3). We know that the inscribed radius (r_i) of any regular n-gon with each side a is given as follows

$$\boldsymbol{r_i = inscribed\ radius\ of\ any\ regular\ n-gon\ = \frac{a}{2}cot\frac{\pi}{n}}$$

$$\therefore\ O_1T = inscribed\ radius\ of\ square = \frac{a}{2}cot\frac{\pi}{4} = \frac{a}{2}\ \ \&$$

$$\therefore\ O_2T = inscribed\ radius\ of\ regular\ hexagon = \frac{a}{2}cot\frac{\pi}{6} = \frac{a\sqrt{3}}{2}$$

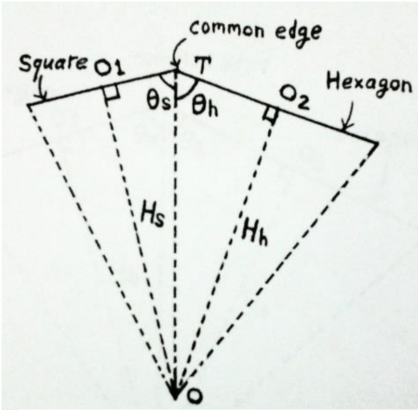

Figure 3: Square face with centre O_1 & regular hexagonal face with centre O_2 with a common edge (denoted by point T) normal to the plane of paper

In right ΔOO_1T

$$tan\theta_s = \frac{OO_1}{O_1T} = \frac{H_s}{\left(\frac{a}{2}\right)} = \frac{\left(\frac{(3+\sqrt{2})a}{2}\right)}{\left(\frac{a}{2}\right)} = (3+\sqrt{2})$$

$$\therefore\ \ \boldsymbol{\theta_s = \tan^{-1}(3+\sqrt{2}) \approx 77.23561032^o} \quad \ldots\ldots\ldots\ldots\ldots\ldots (VIII)$$

In right ΔOO_2T

$$tan\theta_h = \frac{OO_2}{O_2T} = \frac{H_h}{\left(\frac{a\sqrt{3}}{2}\right)} = \frac{\left(\frac{\sqrt{3}(1+\sqrt{2})a}{2}\right)}{\left(\frac{a\sqrt{3}}{2}\right)} = (1+\sqrt{2})$$

$$\therefore \quad \boldsymbol{\theta_h = \tan^{-1}(1+\sqrt{2}) = 67.5^o} \qquad \dots\dots\dots\dots\dots\dots (IX)$$

$$\Rightarrow \theta_s + \theta_h = \tan^{-1}(3+\sqrt{2}) + \tan^{-1}(1+\sqrt{2}) = \tan^{-1}\left(\frac{(3+\sqrt{2})+(1+\sqrt{2})}{1-(3+\sqrt{2})(1+\sqrt{2})}\right) = \tan^{-1}\left(\frac{4+2\sqrt{2}}{1-(5+4\sqrt{2})}\right)$$

$$= \tan^{-1}\left(\frac{-(4+2\sqrt{2})}{4+4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{4+2\sqrt{2}}{4+4\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{1+\sqrt{2}}{2+\sqrt{2}}\right) = \pi - \tan^{-1}\left(\frac{(1+\sqrt{2})(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}\right)$$

$$= \pi - \tan^{-1}\left(\frac{\sqrt{2}}{2}\right) = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

Hence**, dihedral angle between the square face & the regular hexagonal face is given as**

$$\boldsymbol{\theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 144.7356103^o}$$

b. Angle between square face & regular octagonal face: Draw the perpendiculars OO_1 & OO_3 from the centre O of great rhombicuboctahedron to the square face & the regular octagonal face which have a common edge (See figure 4).

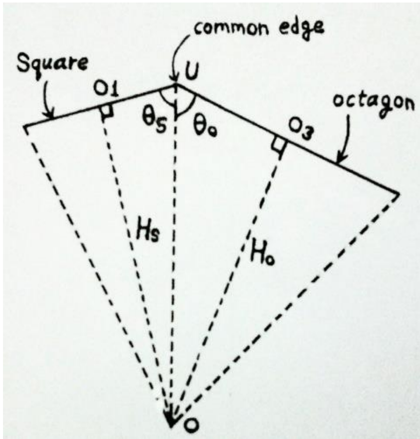

Figure 4: Square face with centre O_1 & regular octagonal face with centre O_3 with a common edge (denoted by point U) normal to the plane of paper

$$O_3U = inscribed\ radius\ of\ regular\ octagon = \frac{a}{2}cot\frac{\pi}{8} = \frac{a(1+\sqrt{2})}{2}$$

In right ΔOO_3U

$$tan\theta_o = \frac{OO_3}{O_3U} = \frac{H_o}{\left(\frac{a(1+\sqrt{2})}{2}\right)} = \frac{\left(\frac{(1+2\sqrt{2})a}{2}\right)}{\left(\frac{a(1+\sqrt{2})}{2}\right)} = \frac{(1+2\sqrt{2})}{(1+\sqrt{2})}$$

$$= \frac{(1+2\sqrt{2})(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} = \frac{3-\sqrt{2}}{(2-1)} = 3-\sqrt{2}$$

$$\therefore \quad \boldsymbol{\theta_o = \tan^{-1}(3-\sqrt{2}) \approx 57.76438969^o} \qquad \dots\dots\dots\dots\dots\dots (X)$$

$$\Rightarrow \theta_s + \theta_o = \tan^{-1}(3+\sqrt{2}) + \tan^{-1}(3-\sqrt{2}) = \tan^{-1}\left(\frac{(3+\sqrt{2})+(3-\sqrt{2})}{1-(3+\sqrt{2})(3-\sqrt{2})}\right) = \tan^{-1}\left(\frac{6}{1-7}\right)$$

$$= \tan^{-1}\left(\frac{-6}{6}\right) = \tan^{-1}(-1) = \pi - \tan^{-1}(1) = 180^o - 45^o = 135^o$$

Hence**, dihedral angle between the square face & the regular octagonal face is given as**

$$\boldsymbol{\theta_s + \theta_o = 135^o}$$

c. Angle between regular hexagonal face & regular octagonal face: Draw the perpendiculars OO_2 & OO_3 from the centre O of great rhombicuboctahedron to the regular hexagonal face & the regular octagonal face which have a common edge (See figure 5). Now from eq(IX) & (X), we get

$$\theta_h + \theta_o = \tan^{-1}(1+\sqrt{2}) + \tan^{-1}(3-\sqrt{2}) = \tan^{-1}\left(\frac{(1+\sqrt{2})+(3-\sqrt{2})}{1-(1+\sqrt{2})(3-\sqrt{2})}\right)$$

$$= \tan^{-1}\left(\frac{4}{1-(1+2\sqrt{2})}\right) = \tan^{-1}\left(\frac{-4}{2\sqrt{2}}\right) = \tan^{-1}(-\sqrt{2}) = \pi - \tan^{-1}(\sqrt{2})$$

Hence**, dihedral angle between the regular hexagonal face & the regular octagonal face is given as**

$$\boldsymbol{\theta_h + \theta_o = \pi - \tan^{-1}(\sqrt{2}) \approx 125.2643897^o}$$

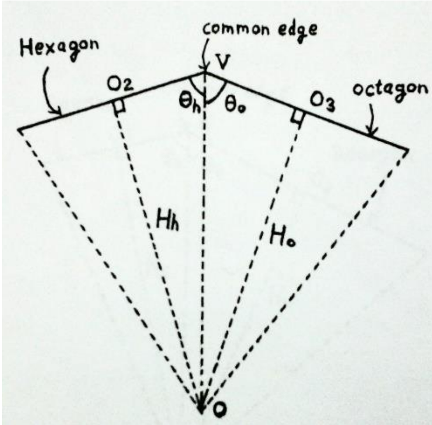

Figure 5: Regular hexagonal face with centre O_2 & regular octagonal face with centre O_3 with a common edge (denoted by point V) normal to the plane of paper

Construction of a solid great rhombicuboctahedron: In order to construct a solid great rhombicuboctahedron with edge length a there are two methods

1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Construct 12 congruent right pyramids with square base of side length a & normal height (H_s)

$$H_s = \frac{(3+\sqrt{2})a}{2} \approx 2.207106781a$$

Construct 8 congruent right pyramids with regular hexagonal base of side length a & normal height (H_h)

$$H_h = \frac{\sqrt{3}(1+\sqrt{2})a}{2} \approx 2.090770275a$$

Construct 6 congruent right pyramids with regular octagonal base of side length a & normal height (H_o)

$$H_o = \frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$$

Now, paste/bond by joining all these elementary right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that 4 edges of each square base (face) coincide with the edges of 2 regular hexagonal bases & 2 regular octagonal bases (faces). Thus a solid great rhombicuboctahedron, with 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of edge length a, is obtained.

2. Facing a solid sphere: It is a method of facing, first we select a **blank as a solid sphere** of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of a great rhombicuboctahedron. Then, we perform the facing operations on the solid sphere to generate 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each of equal edge length.

Let there be a blank as a solid sphere with a diameter D. Then the edge length a, of a great rhombicuboctahedron of the maximum volume to be produced, can be co-related with the diameter D by **relation of outer radius $(\boldsymbol{R_o})$ with edge length $(\boldsymbol{a})$ of the great rhombicuboctahedron** as follows

$$R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}}$$

Now, substituting $R_o = {}^{D}/_{2}$ in the above expression, we have

$$\frac{D}{2} = \frac{a}{2}\sqrt{13+6\sqrt{2}} \quad or \quad a = \frac{D}{\sqrt{13+6\sqrt{2}}}$$

$$\boldsymbol{a = \frac{D}{\sqrt{13+6\sqrt{2}}} \approx 0.215739405D}$$

Above relation is very useful for determining the edge length a of a great rhombicuboctahedron to be produced from a solid sphere with known diameter D for manufacturing purpose.

Hence, the **maximum volume of great rhombicuboctahedron** produced from a solid sphere is given as follows

$$V_{max} = (22+14\sqrt{2})a^3 = (22+14\sqrt{2})\left(\frac{D}{\sqrt{13+6\sqrt{2}}}\right)^3 = \frac{(22+14\sqrt{2})D^3}{(13+6\sqrt{2})\sqrt{13+6\sqrt{2}}}$$

$$= \frac{(22+14\sqrt{2})(13-6\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}} = \frac{(118+50\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}}$$

$$\boldsymbol{V_{max} = \frac{2(59+25\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}} \approx 0.419714736D^3}$$

M**inimum volume of material removed** is given as

$$(V_{removed})_{min} = (volume\ of\ parent\ sphere\ with\ diameter\ D) - (volume\ of\ great\ rhombicuboctahedron)$$

$$= \frac{\pi}{6}D^3 - \frac{2(59+25\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}} = \left(\frac{\pi}{6} - \frac{2(59+25\sqrt{2})D^3}{97\sqrt{13+6\sqrt{2}}}\right)D^3$$

$$\boldsymbol{(V_{removed})_{min} = \left(\frac{\pi}{6} - \frac{2(59+25\sqrt{2})}{97\sqrt{13+6\sqrt{2}}}\right)D^3 \approx 0.103884038D^3}$$

Percentage $(\%)$ of minimum volume of material removed

$$\%\ \boldsymbol{of}\ \boldsymbol{V_{removed}} = \frac{minimum\ volume\ removed}{total\ volume\ of\ sphere} \times 100$$

$$= \frac{\left(\frac{\pi}{6} - \frac{2(59+25\sqrt{2})}{97\sqrt{13+6\sqrt{2}}}\right)D^3}{\frac{\pi}{6}D^3} \times 100 = \left(\mathbf{1} - \frac{12(59+25\sqrt{2})}{97\pi\sqrt{13+6\sqrt{2}}}\right) \times \mathbf{100} \approx \mathbf{19.84\%}$$

It's obvious that when a great rhombicuboctahedron of the maximum volume is produced from a solid sphere then about 19.84% volume of material is removed as scraps. Thus, we can select optimum diameter of blank as a solid sphere to produce a solid great rhombicuboctahedron of the maximum volume (or with maximum desired edge length)

Conclusions: Let there be any great rhombicuboctahedron having 12 congruent square faces, 8 congruent regular hexagonal faces & 6 congruent regular octagonal faces each with edge length a then all its important parameters are calculated/determined as tabulated below

Congruent polygonal faces	**No. of faces**	**Normal distance of each face from the centre of the great rhombicuboctahedron**	**Solid angle subtended by each face at the centre of the great rhombicuboctahedron**
Square	12	$\frac{(3+\sqrt{2})a}{2} \approx 2.207106781a$	$4\sin^{-1}\left(\frac{2-\sqrt{2}}{12}\right) \approx 0.195339779\ sr$
Regular hexagon	8	$\frac{\sqrt{3}(1+\sqrt{2})a}{2} \approx 2.090770275a$	$2\pi - 12\sin^{-1}\left(\frac{\sqrt{2+\sqrt{2}}}{4}\right) \approx 0.5210126\ sr$
Regular octagon	6	$\frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$	$2\pi - 16\sin^{-1}\left(\frac{3-\sqrt{2}}{2\sqrt{6}}\right) \approx 1.009032076\ sr$

Inner (inscribed) radius (R_i)	$R_i = \frac{(1+2\sqrt{2})a}{2} \approx 1.914213562a$
Outer (circumscribed) radius (R_o)	$R_o = \frac{a}{2}\sqrt{13+6\sqrt{2}} \approx 2.317610913a$
Mean radius (R_m)	$R_m = a\left(\frac{3(11+7\sqrt{2})}{2\pi}\right)^{\frac{1}{3}} \approx 2.15290926a$
Surface area (A_s)	$A_s = 12(2+\sqrt{2}+\sqrt{3})a^2 \approx 61.75517244a^2$
Volume (V)	$V = (22+14\sqrt{2})a^3 \approx 41.79898987a^3$

Table for the dihedral angles between the adjacent faces of a great rhombicuboctahedron

Pair of the adjacent faces with a common edge	Square & regular hexagon	Square & regular octagon	Regular hexagon & regular octagon
Dihedral angle of the corresponding pair (of the adjacent faces)	$\theta_s + \theta_h = \pi - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) \approx 144.7356103^o$	$\theta_s + \theta_o = 135^o$	$\theta_h + \theta_o = \pi - \tan^{-1}(\sqrt{2}) \approx 125.2643897^o$

Note: Above articles had been developed & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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