Mathematical Analysis of Trapezohedron With Right Kite Faces

(*Application of HCR's Theory of Polygon*)

Mr Harish Chandra Rajpoot

M.M.M. University of Technology, Gorakhpur-273010 (UP), India ***April, 2015***

Introduction: We are here to analyse an n-gonal trapezohedron/deltohedron having 2n congruent faces each as a **right kite** (**i.e. cyclic quadrilateral consisting of two congruent right triangles with a hypotenuse in common**), 4n edges & $(2n+2)$ vertices lying on a spherical surface with a certain radius. All 2n right kite faces are at an equal normal distance from the centre of the trapezohedron. Each of 2n right kite faces always has two right angles, one acute angle (α) & other obtuse angle (β) $(\forall\, \beta = 180^o - \alpha)$**.** Its each face has two pairs of unequal sides (edges) $a\,\&\, b$ $(\forall\, a \leq b,$ $equality\ holds\ only\ in\ case\ of\ a\ cube)$ & can be divided into two congruent right triangles having longer diagonal of the face as their common hypotenuse. It has two identical & diagonally opposite vertices say vertices C & E out of total (2n+2) vertices at each of which n right kite faces meet together & rest 2n vertices are identical at each of which three right kite faces meet together (See the figure 1).

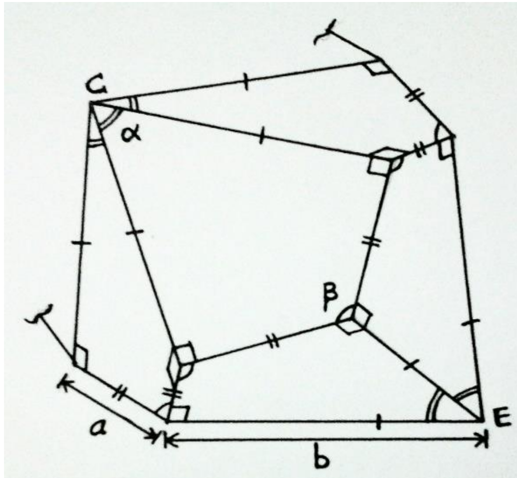

Figure 1: A trapezohedron having 2n+2 vertices, 4n edges & 2n congruent faces each as a right kite having two pairs of unequal sides $a\,\&\, b$ $(\forall\, a < b)$, two right angles, one acute angle α & one obtuse angle β $(\forall\, \beta = 180^o - \alpha)$

$$no.\,of\ congruent\ right\ kite\ faces = 2n$$

$$no.\,of\ edges = 4n \quad \& \quad no.\,of\ vertices = 2n + 2$$

Analysis of n-gonal trapezohedron/deltohedron: Let there be a **trapezohedron** having 2n congruent faces each as a **right kite having two pairs of unequal sides (edges)** $a\,\&\, b$ $(\forall\, a < b)$. Now let's first determine the relation between two unequal sides of the right kite face of the trapezohedron by calculating the ratio of unequal sides (edges) a & b in the generalized form.

Derivation of relation between unequal sides (edges) $a\,\&\, b$ of each right kite face of the trapezohedron/deltohedron: Let h be the normal distance of each of 2n congruent right kite faces from the centre O of a trapezohedron. Now draw a perpendicular OO' from the centre O of the polyhedron at the point O' to any of its right kite faces say face ABCD. Since the right kite face ABCD is a cyclic quadrilateral hence all its vertices A, B, C & D lie on the circle & the perpendicular OO' will have its foot O' at the centre of the circumscribed circle. (See figure 2). Now join all the vertices A, B, C & D to the centre O' to obtain two congruent right triangles $\Delta ABC\ \&\ \Delta\, ADC$. Thus we have,

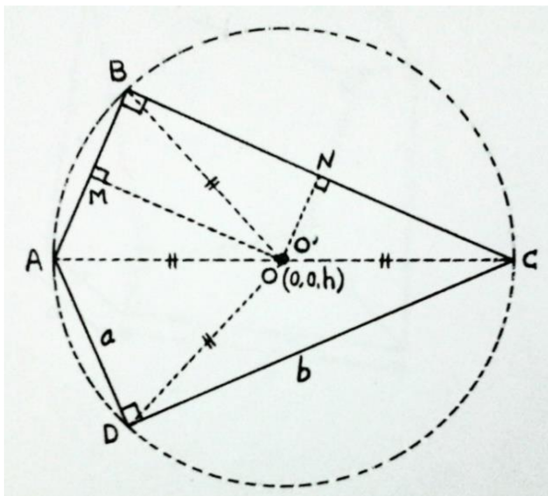

Figure 2: A perpendicular OO' (normal to the plane of paper) is drawn from the centre $O\,(0,0,h)$ of the trapezohedron to the right kite face ABCD at the circumscribed centre O' of face ABCD with $AB = AD = a\,\&\, BC = CD = b\ \ \forall\, a < b$

$$AB = AD = a\ \&\ BC = CD = b \quad \forall\, a < b$$

$$\Rightarrow\ O'A = O'B = O'C = O'D = \frac{AC}{2} = \frac{\sqrt{(AB)^2 + (BC)^2}}{2} = \frac{\sqrt{a^2 + b^2}}{2}$$

Now, draw the perpendiculars O'M & O'N to the sides AB & BC at their mid-points M & N respectively. Thus isosceles $\Delta AO'B$ is divided into two congruent right triangles $\Delta O'MA$ & $\Delta O'MB$. Similarly, isosceles $\Delta BO'C$ is divided into two congruent right triangles $\Delta O'NB$ & $\Delta O'NC$.

In right $\Delta O'MA$

$$\boldsymbol{O'M} = \sqrt{(O'A)^2 - (AM)^2} = \sqrt{\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 - \left(\frac{a}{2}\right)^2} = \frac{\boldsymbol{b}}{\boldsymbol{2}}$$

Similarly, in right $\Delta O'NB$

$$\boldsymbol{O'N} = \sqrt{(O'B)^2 - (BN)^2} = \sqrt{\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2 - \left(\frac{b}{2}\right)^2} = \frac{\boldsymbol{a}}{\boldsymbol{2}}$$

We know from **HCR's Theory of Polygon** that the **solid angle** $(\boldsymbol{\omega})$**, subtended by a right triangle having orthogonal sides** $\boldsymbol{a}$ & $\boldsymbol{b}$ **at any point at a normal distance** $\boldsymbol{h}$ **on the vertical axis passing through the common vertex of the side b & hypotenuse,** is given by **HCR's Standard Formula-1** as follows

$$\boldsymbol{\omega = \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left\{\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{h}{\sqrt{h^2+b^2}}\right)\right\}}$$

Hence, **solid angle** $(\boldsymbol{\omega_{\Delta O'MA}})$ subtended by **right** $\boldsymbol{\Delta O'MA}$ at the centre O $(0,0,h)$ of polyhedron is given as

$$\omega_{\Delta O'MA} = \sin^{-1}\left(\frac{(AM)}{\sqrt{(AM)^2+(O'M)^2}}\right) - \sin^{-1}\left\{\left(\frac{(AM)}{\sqrt{(AM)^2+(O'M)^2}}\right)\left(\frac{(OO')}{\sqrt{(OO')^2+(O'M)^2}}\right)\right\}$$

Now, by substituting all the corresponding values in the above expression we have

$$\omega_{\Delta O'MA} = \sin^{-1}\left(\frac{\left(\frac{a}{2}\right)}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{\left(\frac{a}{2}\right)}{\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2+\left(\frac{b}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left\{\left(\frac{a}{\sqrt{a^2+b^2}}\right)\left(\frac{2h}{\sqrt{4h^2+b^2}}\right)\right\}$$

$$\therefore\ \boldsymbol{\omega_{\Delta O'MA} = \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right)} \quad \ldots\ldots\ldots\ldots.(I)$$

Similarly, **solid angle** $(\boldsymbol{\omega_{\Delta O'NB}})$ subtended by **right** $\boldsymbol{\Delta O'NB}$ at the centre O $(0,0,h)$ of polyhedron is given as

$$\omega_{\Delta O'NB} = \sin^{-1}\left(\frac{(BN)}{\sqrt{(BN)^2+(O'N)^2}}\right) - \sin^{-1}\left\{\left(\frac{(BN)}{\sqrt{(BN)^2+(O'N)^2}}\right)\left(\frac{(OO')}{\sqrt{(OO')^2+(O'N)^2}}\right)\right\}$$

$$\omega_{\Delta O'NB} = \sin^{-1}\left(\frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2+\left(\frac{a}{2}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{\left(\frac{b}{2}\right)}{\sqrt{\left(\frac{b}{2}\right)^2+\left(\frac{a}{2}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2+\left(\frac{a}{2}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{a^2+b^2}}\right)\left(\frac{2h}{\sqrt{4h^2+a^2}}\right)\right\}$$

$$\therefore\ \boldsymbol{\omega_{\Delta O'NB} = \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right)} \quad \dots\dots\dots\dots\dots (II)$$

Now, solid angle (ω_{ABCD}) subtended by the right kite face ABCD at the centre O $(0, 0, h)$ of the polyhedron is given as

$$\boldsymbol{\omega_{ABCD}} = \omega_{\Delta ABC} + \omega_{\Delta ADC} = 2(\omega_{\Delta ABC}) \quad (since,\ \Delta ABC\ \&\Delta ADC\ are\ congruents)$$

$$= 2(\omega_{\Delta AO'B} + \omega_{\Delta BO'C}) = 2\{(\omega_{\Delta O'MA} + \omega_{\Delta O'MB}) + (\omega_{\Delta O'NB} + \omega_{\Delta O'NC})\}$$

$$= 2\{2(\omega_{\Delta O'MA}) + 2(\omega_{\Delta O'NB})\} = \boldsymbol{4(\omega_{\Delta O'MA} + \omega_{\Delta O'NB})} \quad (by\ congruent\ triangles)$$

Since all 2n right kite faces of polyhedron are congruent hence the **solid angle subtended by each right kite face at the centre of trapezohedron** is

$$= \frac{total\ solid\ angle}{no. of\ congruent\ right\ kite\ faces} = \frac{4\pi}{2n} = \frac{2\pi}{n}$$

$$\Rightarrow\ \boldsymbol{\omega_{ABCD}} = \boldsymbol{\frac{2\pi}{n}} \quad or \quad 4(\omega_{\Delta O'MA} + \omega_{\Delta O'NB}) = \frac{2\pi}{n} \quad or \quad \omega_{\Delta O'MA} + \omega_{\Delta O'NB} = \frac{2\pi}{4n} = \frac{\pi}{2n}$$

$$\therefore\ \boldsymbol{\omega_{\Delta O'MA} + \omega_{\Delta O'NB} = \frac{\pi}{2n}}$$

Now, by setting the corresponding values from the eq(I) & (II) in the above expression, we get

$$\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right) + \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right) = \frac{\pi}{2n}$$

$$\Rightarrow \left[\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\right) + \sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)\right] - \left[\sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right) + \sin^{-1}\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right)\right] = \frac{\pi}{2n}$$

$$\Rightarrow \left[\sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}}\sqrt{1-\left(\frac{b}{\sqrt{a^2+b^2}}\right)^2} + \frac{b}{\sqrt{a^2+b^2}}\sqrt{1-\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2}\right)\right] - \left[\sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\sqrt{1-\left(\frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\right)^2} + \frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\sqrt{1-\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\right)^2}\right)\right] = \frac{\pi}{2n}$$

$$\Rightarrow \sin^{-1}\left(\frac{a}{\sqrt{a^2+b^2}} \times \frac{a}{\sqrt{a^2+b^2}} + \frac{b}{\sqrt{a^2+b^2}} \times \frac{b}{\sqrt{a^2+b^2}}\right) - \sin^{-1}\left(\frac{2ah}{\sqrt{(a^2+b^2)(4h^2+b^2)}}\sqrt{\frac{4a^2h^2+4b^2h^2+a^4+a^2b^2-4b^2h^2}{(a^2+b^2)(4h^2+a^2)}} + \frac{2bh}{\sqrt{(a^2+b^2)(4h^2+a^2)}}\sqrt{\frac{4a^2h^2+4b^2h^2+a^2b^2+b^4-4a^2h^2}{(a^2+b^2)(4h^2+b^2)}}\right) = \frac{\pi}{2n}$$

$$\Rightarrow \sin^{-1}\left(\frac{a^2}{a^2+b^2}+\frac{b^2}{a^2+b^2}\right)$$
$$-\sin^{-1}\left(\frac{2ah\sqrt{a^2(4h^2+a^2+b^2)}}{(a^2+b^2)\sqrt{(4h^2+a^2)(4h^2+b^2)}}+\frac{2bh\sqrt{b^2(4h^2+a^2+b^2)}}{(a^2+b^2)\sqrt{(4h^2+a^2)(4h^2+b^2)}}\right)=\frac{\pi}{2n}$$

$$\Rightarrow \sin^{-1}\left(\frac{a^2+b^2}{a^2+b^2}\right)-\sin^{-1}\left(\frac{2h(a^2+b^2)\sqrt{(4h^2+a^2+b^2)}}{(a^2+b^2)\sqrt{(4h^2+a^2)(4h^2+b^2)}}\right)=\frac{\pi}{2n}$$

$$\Rightarrow \sin^{-1}(1)-\sin^{-1}\left(2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}}\right)=\frac{\pi}{2n}$$

$$\Rightarrow \sin^{-1}\left(2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}}\right)=\sin^{-1}(1)-\frac{\pi}{2n}=\frac{\pi}{2}-\frac{\pi}{2n}$$

$$\Rightarrow 2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}}=sin\left(\frac{\pi}{2}-\frac{\pi}{2n}\right)=cos\frac{\pi}{2n}$$

$$\Rightarrow \left(2h\sqrt{\frac{4h^2+a^2+b^2}{(4h^2+a^2)(4h^2+b^2)}}\right)^2=\left(cos\frac{\pi}{2n}\right)^2=cos^2\frac{\pi}{2n}$$

$$\Rightarrow \frac{4h^2(4h^2+a^2+b^2)}{(4h^2+a^2)(4h^2+b^2)}=cos^2\frac{\pi}{2n}$$

$$\Rightarrow 16h^4+4(a^2+b^2)h^2=(16h^4+4(a^2+b^2)h^2+a^2b^2)cos^2\frac{\pi}{2n}$$

$$\Rightarrow 16\left(1-cos^2\frac{\pi}{2n}\right)h^4+4(a^2+b^2)\left(1-cos^2\frac{\pi}{2n}\right)h^2-a^2b^2cos^2\frac{\pi}{2n}=0$$

$$\Rightarrow 16sin^2\frac{\pi}{2n}h^4+4sin^2\frac{\pi}{2n}(a^2+b^2)h^2-a^2b^2cos^2\frac{\pi}{2n}=0$$

$$\Rightarrow \boldsymbol{16h^4+4(a^2+b^2)h^2-a^2b^2cot^2\frac{\pi}{2n}=0}$$

Now, solving the above bi-quadratic equation to obtain the values of h^2 as follows

$$\Rightarrow h^2=\frac{-4(a^2+b^2)\pm\sqrt{\left(-4(a^2+b^2)\right)^2+4(16)a^2b^2cot^2\frac{\pi}{2n}}}{2(16)}$$
$$=\frac{-4(a^2+b^2)\pm 4\sqrt{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}}}{32}$$

$$=\frac{-(a^2+b^2)\pm\sqrt{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}}}{8}$$

Since, $h>0\ or\ h^2>0$ hence, by taking positive sign we get the required value of normal height h as follows

$$h^2=\frac{-(a^2+b^2)+\sqrt{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}}}{8}=\frac{\sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}-(a^2+b^2)}{8}$$

$$\boldsymbol{Normal\ distance,\ h=\frac{\sqrt{\sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}-(a^2+b^2)}}{2\sqrt{2}}}\ \ \ldots\ldots\ldots\ldots\ldots..(III)$$

Outer (circumscribed) radius (R_o) of n-gonal trapezohedron (i.e. radius of the spherical surface passing through all $(2n+2)$ vertices of trapezohedron): Let R_o be the circumscribed radius i.e. the radius of the spherical surface passing through all 2n+2 vertices of a trapezohedron. (See figure 3 below)

Consider the right kite face ABCD & join all its vertices A, B, C & D to the centre O of n-gonal trapezohedron. Draw a perpendicular OO' from the centre O to the face ABCD at the point O'. Since the spherical surface with a radius R_o is passing though all 2n+2 vertices of polyhedron hence we have

$$OA=OB=OC=OD=R_o\ \ \&\ \ OO'=h$$

Now, in right $\Delta OO'A$

$$\Rightarrow\ (OA)^2=(OO')^2+(O'A)^2=h^2+(O'A)^2$$

$$=\left(\frac{\sqrt{\sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}-(a^2+b^2)}}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2$$

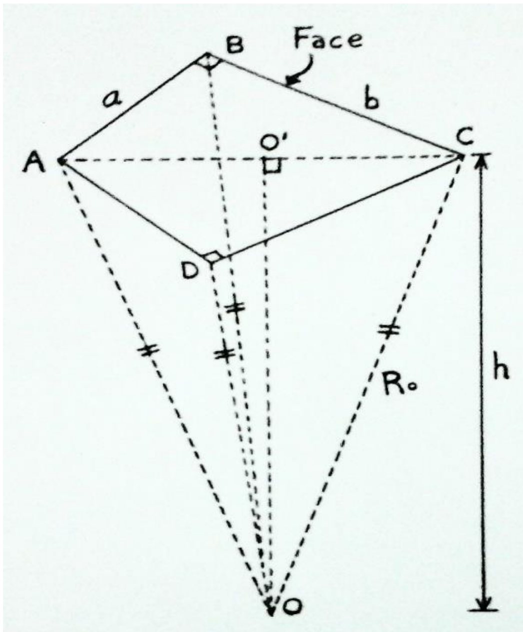

Figure 3: An elementary right pyramid OABCD is obtained by joining all four vertices A, B, C & D of right kite face ABCD to the centre O of an n-gonal trapezohedron/deltohedron

$$=\frac{\sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}-(a^2+b^2)}{8}+\frac{a^2+b^2}{4}$$

$$=\frac{\sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}-(a^2+b^2)+2(a^2+b^2)}{8}$$

$$\Rightarrow\ {R_o}^2=\frac{\sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}+a^2+b^2}{8}$$

$$\Rightarrow\ \boldsymbol{8{R_o}^2=\sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}+a^2+b^2}\ \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(IV)$$

Since, all 12 vertices are located on the circumscribed spherical surface, **Let us consider a great circles with the centre 'O' passing through two identical & diagonally opposite vertices C & E** (as shown in the figure 4 below). Hence the line CE is a diametric line passing through the centre O of a great circle (on the **circumscribed spherical surface**) & the vertices C, A & E are lying on the (great) circle hence, the angle $CAE=90^o$. Now

In right ΔCAE

$$(CE)^2 = (AC)^2 + (AE)^2$$

Now, by substituting all the corresponding values in the above expression we get

$$(2R_o)^2 = \left(\sqrt{a^2+b^2}\right)^2 + (b)^2$$

$$4{R_o}^2 = a^2+b^2+b^2 = a^2+2b^2$$

$$\Rightarrow \ \mathbf{8{R_o}^2 = 2a^2+4b^2} \qquad \dots\dots\dots\dots\dots..(V)$$

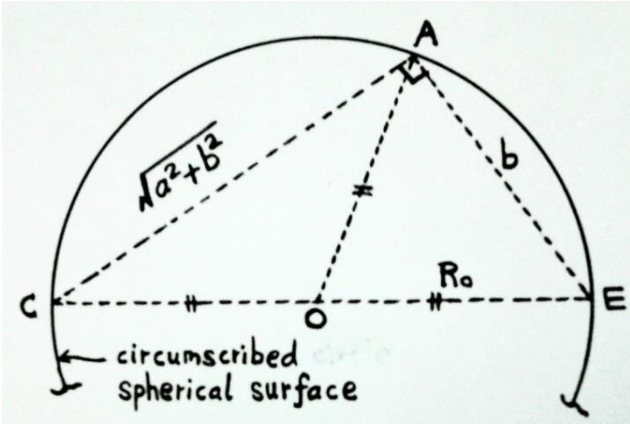

Figure 4: Two vertices C & E are identical & diagonally opposite. Vertices C, A & E are lying on a great circle on the circumscribed spherical surface & angle $CAE = 90^o$

Now, equating eq(IV) & (V), we get

$$2a^2+4b^2 = \sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}+a^2+b^2$$

$$\Rightarrow \ a^2+3b^2 = \sqrt{\left\{(a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}\right\}}$$

$$\Rightarrow \ (a^2+3b^2)^2 = (a^2+b^2)^2+4a^2b^2cot^2\frac{\pi}{2n}$$

$$\Rightarrow \ a^4+9b^4+6a^2b^2 = a^4+b^4+2a^2b^2+4a^2b^2cot^2\frac{\pi}{2n}$$

$$\Rightarrow \ 8b^4+4a^2b^2-4a^2b^2cot^2\frac{\pi}{2n} = 0 \ \ or \ \ 8b^4-4\left(cot^2\frac{\pi}{2n}-1\right)a^2b^2 = 0$$

$$or \ \ 2b^4-\left(cot^2\frac{\pi}{2n}-1\right)a^2b^2 = 0 \ \ \Rightarrow \ b^2\left(2b^2-\left(cot^2\frac{\pi}{2n}-1\right)a^2\right) = 0$$

But, $b \neq 0$ hence, we have

$$2b^2-\left(cot^2\frac{\pi}{2n}-1\right)a^2 = 0 \ \ or \ \ \frac{a^2}{b^2} = \frac{2}{cot^2\frac{\pi}{2n}-1} \ \ or \ \ \frac{a}{b} = \sqrt{\frac{2}{cot^2\frac{\pi}{2n}-1}}$$

$$\frac{a}{b} = \sqrt{\frac{2tan^2\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}} = \sqrt{tan\frac{\pi}{2n}\left(\frac{2tan\frac{\pi}{2n}}{\sqrt{1-tan^2\frac{\pi}{2n}}}\right)} = \sqrt{tan\frac{\pi}{2n}\left(tan\frac{\pi}{n}\right)}$$

$$= \sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}} \quad or \quad a = b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}$$

$\therefore$ ***The required relation between unequal sides (edges) a & b of n gonal trapezohedron***

$$\boldsymbol{a = b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}} \qquad \left(\forall \ \ n \in N \ \& \ \ n \geq 3 \ \ \Rightarrow a \leq b\right)}$$

In right ΔABC (from the figure2)

$$tan\alpha ACB = \frac{AB}{BC} = \frac{a}{b} \ \Rightarrow \ tan\frac{\alpha}{2} = \frac{a}{b} = \sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}} \quad or \ \ \alpha = 2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)$$

$$\therefore \ \boldsymbol{acute\ angle, \alpha = 2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)} \quad \forall\ \boldsymbol{n \geq 3 \Rightarrow \alpha \leq 90^o}$$

$$\therefore \ \boldsymbol{obtuse\ angle, \beta = 180^o - \alpha}$$

Now, setting the value of smaller side (edge) a in term of larger side (edge) b in the eq(III), we get

$$h = \frac{\sqrt{\sqrt{\left\{(a^2+b^2)^2 + 4a^2b^2cot^2\frac{\pi}{2n}\right\}} - (a^2+b^2)}}{2\sqrt{2}}$$

$$= \frac{\sqrt{\sqrt{\left\{\left(\left(b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)^2 + b^2\right)^2 + 4\left(b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)^2 b^2cot^2\frac{\pi}{2n}\right\}} - \left(\left(b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)^2 + b^2\right)}}{2\sqrt{2}}$$

$$= \frac{\sqrt{\sqrt{\left\{\left(b^2tan\frac{\pi}{n}tan\frac{\pi}{2n} + b^2\right)^2 + 4b^4tan\frac{\pi}{n}tan\frac{\pi}{2n}cot^2\frac{\pi}{2n}\right\}} - \left(b^2tan\frac{\pi}{n}tan\frac{\pi}{2n} + b^2\right)}}{2\sqrt{2}}$$

$$= \frac{\sqrt{\sqrt{b^4\left\{\left(1 + tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^2 + 4tan\frac{\pi}{n}cot\frac{\pi}{2n}\right\}} - b^2\left(1 + tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}}{2\sqrt{2}}$$

$$= \frac{\sqrt{b^2\sqrt{\left\{\left(1 + \frac{2tan\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}tan\frac{\pi}{2n}\right)^2 + 4\frac{2tan\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}cot\frac{\pi}{2n}\right\}} - b^2\left(1 + \frac{2tan\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}tan\frac{\pi}{2n}\right)}}{2\sqrt{2}}$$

$$= \frac{b\sqrt{\sqrt{\left\{\left(\frac{1+tan^2\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}\right)^2 + \frac{8}{1-tan^2\frac{\pi}{2n}}\right\}} - \left(\frac{1+tan^2\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}\right)}}{2\sqrt{2}}$$

$$= \frac{b\sqrt{\sqrt{\left\{\frac{\left(1+tan^2\frac{\pi}{2n}\right)^2 + 8\left(1-tan^2\frac{\pi}{2n}\right)}{\left(1-tan^2\frac{\pi}{2n}\right)^2}\right\}} - \left(\frac{1+tan^2\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}\right)}}{2\sqrt{2}}$$

$$= \frac{b\sqrt{\frac{1}{1-tan^2\frac{\pi}{2n}}\sqrt{\left\{1 + tan^4\frac{\pi}{2n} + 2tan^2\frac{\pi}{2n} + 8 - 8tan^2\frac{\pi}{2n}\right\}} - \left(\frac{1+tan^2\frac{\pi}{2n}}{1-tan^2\frac{\pi}{2n}}\right)}}{2\sqrt{2}}$$

$$=\frac{b\sqrt{\frac{1}{1-\tan^2\frac{\pi}{2n}}\sqrt{\left\{9+\tan^4\frac{\pi}{2n}-6\tan^2\frac{\pi}{2n}\right\}}-\left(\frac{1+\tan^2\frac{\pi}{2n}}{1-\tan^2\frac{\pi}{2n}}\right)}}{2\sqrt{2}}$$

$$=\frac{b\sqrt{\frac{1}{1-\tan^2\frac{\pi}{2n}}\left(\sqrt{\left(3-\tan^2\frac{\pi}{2n}\right)^2}-\left(1+\tan^2\frac{\pi}{2n}\right)\right)}}{2\sqrt{2}}=\frac{b\sqrt{\frac{\left(\left(3-\tan^2\frac{\pi}{2n}\right)-\left(1+\tan^2\frac{\pi}{2n}\right)\right)}{1-\tan^2\frac{\pi}{2n}}}}{2\sqrt{2}}$$

$$=\frac{b\sqrt{\frac{2\left(1-\tan^2\frac{\pi}{2n}\right)}{1-\tan^2\frac{\pi}{2n}}}}{2\sqrt{2}}=\frac{b\sqrt{2}}{2\sqrt{2}}=\frac{b}{2}$$

$$\therefore \boldsymbol{Normal\ distance\ of\ each\ face\ from\ the\ centre\ of\ trapezohedron,}$$

$$\boldsymbol{h=\frac{b}{2}\quad \forall\ a=b\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}}$$

Above is the **required expression to calculate the normal distance h of each right kite face from the centre of n-gonal trapezohedron. Normal height h merely depends on the larger side (edge) b irrespective of no .of faces n. Normal distance h is always equal to the inner (inscribed) radius (R_i) i.e. the radius of the spherical surface touching all 2n congruent right kite faces of n-gonal trapezohedron.**

Now, setting the value of a in term of b in the eq(V), we have

$$8{R_o}^2=2a^2+4b^2\quad\Rightarrow {R_o}^2=\frac{a^2+2b^2}{4}$$

$$\Rightarrow {R_o}^2=\frac{\left(b\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)^2+2b^2}{4}=\frac{b^2\tan\frac{\pi}{n}\tan\frac{\pi}{2n}+2b^2}{4}=\frac{b^2\left(2+\tan\frac{\pi}{n}\tan\frac{\pi}{2n}\right)}{4}$$

$$or\quad R_o=\frac{b}{2}\sqrt{2+\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}$$

$$\therefore \boldsymbol{Outer\ radius\ of\ trapezohedron,\ R_o=\frac{b}{2}\sqrt{2+\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\qquad \forall\ a=b\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}}$$

Above is the **required expression to calculate the outer (circumscribed) radius (R_o) of n-gonal trapezohedron having 2n congruent right kite faces.**

Surface Area (A_s) of n-gonal trapezohedron: Since, each of 2n faces of a uniform polyhedron is a right kite hence the surface area of the trapezohedron is given as

$$A_s=2n\times(Area\ of\ right\ kite\ face)=2n\times\left(2\times(Area\ of\ right\ \Delta ABC)\right)\quad(see\ figure\ 3\ above)$$

$$=4n\times\left(\frac{1}{2}ab\right)=2nb\left(b\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)=2nb^2\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}$$

$$\therefore \boldsymbol{Surface\ area\ of\ trapezohedron,\ A_s=2nb^2\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}}$$

Volume (V) of n-gonal trapezohedron: Since, a trapezohedron has 2n congruent faces each as a right kite hence the **trapezohedron consists of 2n congruent elementary right pyramids each with right kite base (face)**. Hence the volume (V) of trapezohedron is given as (See figure 3 above)

$$V = 2n \times (volume\ of\ elementary\ right\ pyramid\ with\ right\ kite\ face\ ABCD)$$

$$= 2n\left\{\frac{1}{3} \times (area\ of\ right\ kite\ face\ ABCD) \times (normal\ height)\right\} = \frac{2n}{3}(ab) \times (h)$$

$$= \frac{2nb}{3}\left(b\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)\frac{b}{2} = \frac{nb^3}{3}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}$$

$$\therefore \boldsymbol{Volume\ of\ trapezohedron,\ \ V = \frac{nb^3}{3}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}}$$

Mean radius (R_m) of n-gonal trapezohedron: It is the radius of the sphere having a volume equal to that of a given n-gonal trapezohedron/deltohedron. It is calculated as follows

$$volume\ of\ sphere\ with\ mean\ radius\ R_m = volume\ of\ given\ trapezohedron$$

$$\frac{4}{3}\pi(R_m)^3 = \frac{nb^3}{3}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}$$

$$\Rightarrow (R_m)^3 = \frac{nb^3}{4\pi}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}} \quad or\ \ R_m = \left(\frac{nb^3}{4\pi}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)^{\frac{1}{3}} = b\left(\frac{n}{4\pi}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)^{\frac{1}{3}}$$

$$\therefore \boldsymbol{R_m = b\left(\frac{n}{4\pi}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)^{\frac{1}{3}}}$$

For finite values of a or b (known value) $\Rightarrow R_i < R_m < R_o$

We know that the n-gonal trapezohedron with right kite faces has two types of identical vertices. It has two identical & diagonally opposite vertices at each of which n no. of right kite faces meet together & rest 2n are identical at each of which three right kite faces meet together. Thus we would analyse two cases to calculate solid angle subtended by the solid at its two dissimilar vertices by assuming that the eye of the observer is located at any of two dissimilar vertices & directed (focused) to the centre of the n-gonal trapezohedron. Thus let's analyse both the cases as follows

Solid angle subtended by n-gonal trapezohedron at each of its two diagonally opposite vertices:

We know that n no. of congruent right kite faces meet at each of two diagonally opposite vertices of a n-gonal trapezohedron hence by assuming that the eye of the observer is located at one of two identical & diagonally opposite vertices (See figure 5), the solid angle is calculated by using **HCR's standard formula.** According to which**, solid angle (ω), subtended at the vertex (apex point) by a right pyramid with a regular n-gonal base & an angle α between any two consecutive lateral edges meeting at the same vertex,** is mathematically given by the standard (generalized) formula as follows

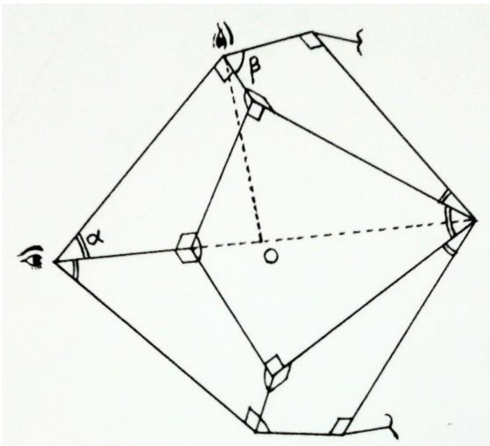

Figure 5: The eye of the observer is located at two different vertices since the trapezohedron has two types of identical vertices

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)} \qquad \forall\ n \in N\ \&\ n \geq 3$$

We know that the acute angle of the right kite face is given as follows

$$\alpha = 2\tan^{-1}\left(\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right) \Rightarrow \tan\frac{\alpha}{2} = \sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}$$

Now, by setting the value of $\tan\frac{\alpha}{2}$ in the above formula, we get solid angle subtended by the (convex) trapezohedron solid at each of two diagonally opposite vertices as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \left(\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)^2}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan\frac{\pi}{n}\left(\tan\frac{\pi}{n} - \tan\frac{\pi}{2n}\right)}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan\frac{\pi}{n}\left(\frac{2\tan\frac{\pi}{2n}}{1-\tan^2\frac{\pi}{2n}} - \tan\frac{\pi}{2n}\right)}\right) \qquad \left(since,\ \ \tan 2x = \frac{2\tan x}{1-\tan^2 x}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan\frac{\pi}{n}\tan\frac{\pi}{2n}\left(\frac{2-1+\tan^2\frac{\pi}{2n}}{1-\tan^2\frac{\pi}{2n}}\right)}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{\cos^2\frac{\pi}{n}\tan\frac{\pi}{n}\tan\frac{\pi}{2n}\left(\frac{1+\tan^2\frac{\pi}{2n}}{1-\tan^2\frac{\pi}{2n}}\right)}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{\cos^2\frac{\pi}{n}\left(\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}\right)\tan\frac{\pi}{2n}\left(\frac{1}{\cos\frac{\pi}{n}}\right)}\right) = 2\pi - 2n\sin^{-1}\left(\sqrt{\sin\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)$$

Hence, the solid angle subtended by the n-gonal trapezohedron at each of two identical & diagonally opposite vertices is given as follows

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\sqrt{\sin\frac{\pi}{n}\tan\frac{\pi}{2n}}\right)} \qquad \left(\forall\ \ \boldsymbol{n} \geq \mathbf{3}\ \&\ \boldsymbol{n}\uparrow\ \Rightarrow \boldsymbol{\omega}\uparrow\right)$$

The above expression shows that the solid angle subtended by the n-gonal trapezohedron at each of two identical & diagonally opposite vertices is independent of the dimensions of n-gonal trapezohedron it depends only on the value of $\boldsymbol{n}$ i.e. no. of congruent right kite faces. Solid angle $(\boldsymbol{\omega})$ increase with the increase in the no. of faces $2\boldsymbol{n}$.

Solid angle subtended by n-gonal trapezohedron at each of 2n identical vertices:

We know that three congruent right kite faces meet together at each of $2n$ identical vertices of an n-gonal trapezohedron hence by assuming that the eye of the observer is located at one of 2n identical vertices (See the upper position of the observer's eye in figure 5 above) the solid angle is calculated by using **formula of tetrahedron** (as discussed in another paper)**.** In this case, three edges meet together at each of 2n identical vertices & make the angles $90^o, 90^o$ & $\beta(=180^o-2\alpha)$ with one another consecutively thus we have

$$\alpha=90^o,\ \ \beta=90^o\ \&\ \gamma=180^o-2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\ \ (\gamma>90^o)$$

Now, let's calculate the constant K by using the formula as follows

$$K=\frac{2sin\frac{\alpha}{2}sin\frac{\beta}{2}sin\frac{\gamma}{2}}{\sqrt{4sin^2\frac{\alpha}{2}sin^2\frac{\beta}{2}-\left(sin^2\frac{\alpha}{2}+sin^2\frac{\beta}{2}-sin^2\frac{\gamma}{2}\right)^2}}$$

$$\boldsymbol{K}=\frac{2sin\frac{90^o}{2}sin\frac{90^o}{2}sin\frac{\left(180^o-2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{2}}{\sqrt{4sin^2\frac{90^o}{2}sin^2\frac{90^o}{2}-\left(sin^2\frac{90^o}{2}+sin^2\frac{90^o}{2}-sin^2\frac{\left(180^o-2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{2}\right)^2}}$$

$$=\frac{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)sin\left(90^o-\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{\sqrt{4\left(\frac{1}{\sqrt{2}}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2-\left(\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2-sin^2\left(90^o-\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)\right)^2}}$$

$$=\frac{cos\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{\sqrt{1-\left(1-cos^2\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)\right)^2}}=\frac{cos\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{\sqrt{1-\left(sin^2\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)\right)^2}}$$

$$=\frac{1}{\sqrt{1+sin^2\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}}=\frac{1}{\sqrt{1+\left\{sin\left(\sin^{-1}\left(\frac{\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{1+\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)^2}}\right)\right)\right\}^2}}$$

$$=\frac{1}{\sqrt{1+\left\{\frac{\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right\}^2}}=\frac{1}{\sqrt{1+\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}}=\sqrt{\frac{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}$$

$$\therefore \boldsymbol{K}=\sqrt{\frac{\mathbf{1}+\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{n}}\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}{\mathbf{1}+\mathbf{2}\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{n}}\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}}\quad \forall\ \boldsymbol{n}\geq \mathbf{3}\ \Rightarrow\ \mathbf{0}<\boldsymbol{K}<\mathbf{1}$$

Now, by substituting all the corresponding values, we get

$$\Rightarrow \boldsymbol{\omega_1}=\mathbf{2}\left[\sin^{-1}\left(\frac{\boldsymbol{sin}\frac{\boldsymbol{\alpha}}{\mathbf{2}}}{\boldsymbol{K}}\right)-\sin^{-1}\left(\boldsymbol{tan}\frac{\boldsymbol{\alpha}}{\mathbf{2}}\sqrt{\left(\frac{\mathbf{1}}{\boldsymbol{K}}\right)^{2}-\mathbf{1}}\right)\right]$$

$$=2\left[\sin^{-1}\left(\frac{sin\frac{90^o}{2}}{\left(\sqrt{\frac{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)}\right)-\sin^{-1}\left(tan\frac{90^o}{2}\sqrt{\left(\frac{1}{\left(\sqrt{\frac{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)}\right)^{2}-1}\right)\right]$$

$$=2\left[\sin^{-1}\left(\frac{1}{\sqrt{2}}\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)-\sin^{-1}\left(\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}-1}\right)\right]$$

$$=2\left[\sin^{-1}\left(\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{2+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)-\sin^{-1}\left(\sqrt{\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right]$$

$$=2\left[\sin^{-1}\left(\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{2+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\sqrt{\frac{1}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}-\sqrt{\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\sqrt{\frac{1}{2+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right]$$

$$=2\left[\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}-\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{2}\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}\right)\right]=2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}-\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{2}\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}\right)$$

Similarly, we can find out the value of solid angle $\boldsymbol{\omega_2}$ (which is equal to $\boldsymbol{\omega_1}$) as follows

$$\Rightarrow \boldsymbol{\omega_2}=\boldsymbol{\omega_1}=\mathbf{2}\sin^{-1}\left(\frac{\sqrt{\mathbf{1}+\mathbf{2}\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{n}}\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}-\sqrt{\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{n}}\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}}{\sqrt{\mathbf{2}}\left(\mathbf{1}+\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{n}}\boldsymbol{tan}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}\right)}\right)$$

$$Now,\ \ \boldsymbol{\omega_3}=2\left[\sin^{-1}\left(\frac{sin\frac{\gamma}{2}}{K}\right)-\sin^{-1}\left(tan\frac{\gamma}{2}\sqrt{\left(\frac{1}{K}\right)^{2}-1}\right)\right]$$

Note: For detailed discussion, go through the paper '**Mathematical Analysis of Tetrahedron by HCR**'

$$= 2\left[\sin^{-1}\left(\frac{sin\frac{\left(180^o - 2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{2}}{\left(\sqrt{\frac{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)}\right)\right.$$

$$\left. - \sin^{-1}\left(tan\frac{\left(180^o - 2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{2}\sqrt{\left(\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)^2 - 1}\right)\right]$$

$$= 2\left[\sin^{-1}\left(cos\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right.$$

$$\left. - \sin^{-1}\left(cot\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}} - 1}\right)\right]$$

$$= 2\left[\sin^{-1}\left(cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right)\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right.$$

$$\left. - \sin^{-1}\left(cot\left(\cot^{-1}\left(\frac{1}{\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right)\sqrt{\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right]$$

$$= 2\left[\sin^{-1}\left(\frac{1}{\sqrt{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\sqrt{\frac{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\sqrt{\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right]$$

$$= 2\left[\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right]$$

$$= 2\left[\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}\sqrt{\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}} - \frac{1}{\sqrt{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\sqrt{\frac{tan^2\frac{\pi}{n}tan^2\frac{\pi}{2n}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^2}}\right)\right]$$

$$= 2\left[\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^{\frac{3}{2}}} - \frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^{\frac{3}{2}}}\right)\right]$$

$$\boldsymbol{\omega_3 = 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}-tan\frac{\pi}{n}tan\frac{\pi}{2n}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^{\frac{3}{2}}}\right)}$$

The largest angle of parametric triangle is C which is calculated by using cosine formula as follows

$$C = \cos^{-1}\left(\frac{sin^2\frac{\alpha}{2}+sin^2\frac{\beta}{2}-sin^2\frac{\gamma}{2}}{2sin\frac{\alpha}{2}sin\frac{\beta}{2}}\right) = \cos^{-1}\left(\frac{sin^2\frac{90^o}{2}+sin^2\frac{90^o}{2}-sin^2\frac{\gamma}{2}}{2sin\frac{90^o}{2}sin\frac{90^o}{2}}\right)$$

$$= \cos^{-1}\left(\frac{\left(\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2-sin^2\frac{\gamma}{2}}{2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)}\right) = \cos^{-1}\left(1-sin^2\frac{\gamma}{2}\right) = \cos^{-1}\left(cos^2\frac{\gamma}{2}\right)$$

$$= \cos^{-1}\left(cos^2\frac{\left(180^o-2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)}{2}\right) = \cos^{-1}\left(sin^2\left(\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right)\right)\right)$$

$$= \cos^{-1}\left(\left(sin\left(\sin^{-1}\left(\frac{\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)\right)\right)^2\right) = \cos^{-1}\left(\left(\frac{\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)^2\right)$$

$$C = \cos^{-1}\left(\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right) \quad \Rightarrow \quad \boldsymbol{0 < \left(\frac{tan\frac{\pi}{n}tan\frac{\pi}{2n}}{1+tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right) < 1 \quad \forall \quad n \geq 3}$$

$$\therefore \boldsymbol{C < 90^o}$$

Hence, the **parametric triangle is an acute angled triangle**.

Hence the **foot of perpendicular (F.O.P.) drawn from the vertex of a tetrahedron to the plane of parametric triangle will lie within its boundary** hence, the **solid angle subtended by the n-gonal trapezohedron at the vertex** is the **sum of the magnitudes of solid angles** as follows

$$\boldsymbol{\omega} = \omega_1 + \omega_2 + \omega_3$$

$$= 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}-\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{2}\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}\right) + 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}-\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{2}\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}\right)$$
$$+ 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}-tan\frac{\pi}{n}tan\frac{\pi}{2n}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^{\frac{3}{2}}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}-\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{2}\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}\right) + 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}-tan\frac{\pi}{n}tan\frac{\pi}{2n}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^{\frac{3}{2}}}\right)$$

Hence, the solid angle subtended by the n-gonal trapezohedron at each of 2n identical vertices (at each of which three right kite faces meet together) is given as follows

$$\omega = 4\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}-\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{2}\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}\right) + 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}-tan\frac{\pi}{n}tan\frac{\pi}{2n}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^{\frac{3}{2}}}\right)$$

The above formula is equally applicable for any arbitrary value of n (i.e. a natural number ≥ 3).

Dihedral angles between the adjacent right kite faces: We know that there are two unequal edges a & b of each of 2n right kite faces which are common (shared) between two adjacent right kite faces. Hence there are two cases of common edge (side) 1. Smaller common edge & 2. Larger common edge. In order to calculate dihedral angles between the adjacent right kite faces with a common edge in an n-gonal trapezohedron, let's consider both the cases one-by-one as follows

a. Dihedral angle between the adjacent right kite faces having smaller common edge:

In this case, consider any two adjacent right kite faces sharing a smaller common edge (as denoted by the point M normal to the plane of paper in the figure 6) each at a normal distance $OO' = h$ from the centre O of the uniform polyhedron. Now

In right $\Delta OO'M$

$$\Rightarrow \tan\angle OMO' = \frac{OO'}{MO'} \Rightarrow \tan\theta = \frac{h}{\left(\frac{b}{2}\right)}$$

$$\theta = \tan^{-1}\left(\frac{\frac{b}{2}}{\frac{b}{2}}\right) = \tan^{-1}(1) = 45^o \quad \left(since,\ h = \frac{b}{2}\right)$$

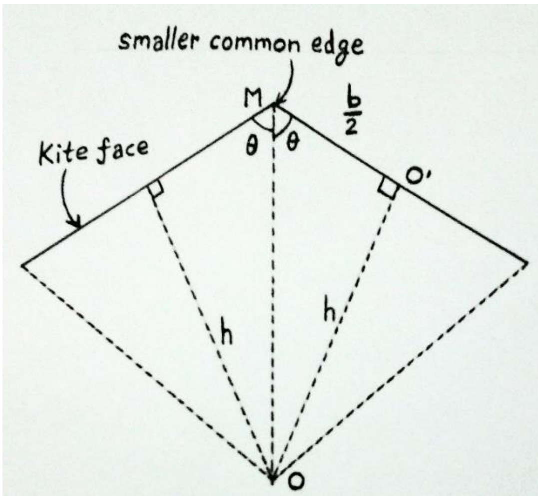

Figure 6: Two adjacent right kite faces sharing a smaller common edge of length a denoted by the point M normal to the plane of paper.

Hence, the dihedral angle between any two adjacent right kite faces meeting at smaller common edge (a) is given as

$$2\theta = 2(45^o) = 90^o$$

Above value of dihedral angle indicates that each two adjacent right kite faces meet at right angle (90^o) at the smaller edge (side) of n-gonal trapezohedron which is independent on the no. of faces ($2n$) & edge lengths a & b in any n-gonal trapezohedron with congruent right kite faces.

b. Dihedral angle between the adjacent right kite faces having larger common edge:

In this case, consider any two adjacent right kite faces meeting at larger common edge (as denoted by the point N normal to the plane of paper in the figure 7 below) each at a normal distance $OO' = h$ from the centre O of the n-gonal trapezohedron. Now

In right $\Delta OO'N$

$$\Rightarrow tan\,\llcorner ONO' = \frac{OO'}{NO'} \Rightarrow\ tan\theta = \frac{h}{\left(\frac{a}{2}\right)}$$

$$\theta = \tan^{-1}\left(\frac{\frac{b}{2}}{\frac{a}{2}}\right) = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{b}{b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right)$$

$$= \tan^{-1}\left(\frac{1}{\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}\right) = \tan^{-1}\left(\sqrt{cot\frac{\pi}{n}cot\frac{\pi}{2n}}\right)$$

Hence, the dihedral angle between any two adjacent right kite faces sharing larger common edge (i.e. b) is given as

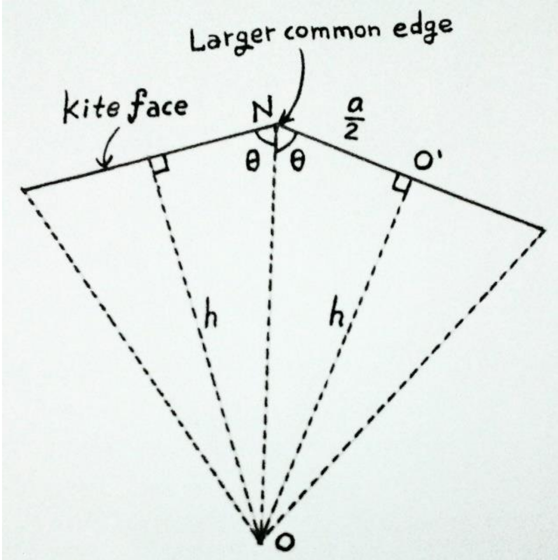

Figure 7: Two adjacent right kite faces sharing a larger common edge of length b denoted by the point N normal to the plane of paper.

$$\boldsymbol{2\theta = 2\tan^{-1}\left(\sqrt{cot\frac{\pi}{n}cot\frac{\pi}{2n}}\right)}$$

Above value of dihedral angle indicates that each two adjacent right kite faces meet at an angle (2θ) at the larger edge (side) of n-gonal trapezohedron which is dependent on the no. of faces $(2n)$ but doesn't depend on the edge lengths a & b in any uniform polyhedron with congruent right kite faces.

Construction of a solid n-gonal trapezohedron: In order to construct a solid n-gonal trapezohedron having 2n congruent faces each as a right kite with two pairs of unequal sides (edges) a & b while one of them is required to be known for calculating important dimensions of polyhedron.

Step 1: First we construct right kite base (face) with the help of known values of a & b while one of these is required to be known while other unknown side (edge) is calculated by the following relations of a & b

$$a = b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}$$

Step 2: Construct all its 2n congruent elementary right pyramids with right kite base (face) of a normal height h given as (See figure 3 above)

$$h = \frac{b}{2} \quad \forall\ \ a = b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}$$

Step 2: paste/bond by joining all 2n elementary right pyramids by overlapping their lateral faces & keeping their apex points coincident with each other such that n right-kite faces meet at **each of two identical & diagonally opposite vertices** and three right kite faces meet at each of rest 2n identical vertices. Thus, a solid n-gonal trapezohedron, with 2n congruent faces each as a right kite with two pairs of unequal sides of a & b, is obtained.

Important deductions: We can analyse any n-gonal trapezohedron having 2n congruent right kite faces by setting values of no. of faces n meeting at each of its two identical & diagonally opposite vertices as follows

1. n-gonal trapezohedron having least no. of right kite faces ($\boldsymbol{n=3}$)

By setting $n=3$ in all above generalized formula of n-gonal trapezohedron, we can calculate various important parameters.

$$no. of\ congruent\ right\ kite\ faces = 2n = 2(3) = 6$$

$$no. of\ edges = 4n = 4(3) = 12 \quad \& \quad no. of\ vertices = 2n + 2 = 2(3) + 2 = 8$$

Now by setting $n=3$, we get ratio between unequal sides (edges) a & b as follows

$$a = b\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}} = b\sqrt{tan\frac{\pi}{3}tan\frac{\pi}{6}} = b\sqrt{(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)} = b\sqrt{1} = b \ \Rightarrow a = b$$

Thus, above result shows that the trapezohedron is a solid having 6 congruent right kite faces

$$\therefore\ acute\ angle, \alpha = 2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}\right) = 2\tan^{-1}\left(\sqrt{tan\frac{\pi}{3}tan\frac{\pi}{6}}\right) = 2\tan^{-1}\left(\sqrt{(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}\right)$$

$$= 2\tan^{-1}(1) = 90^o$$

$$\Rightarrow \boldsymbol{\alpha = \beta = 180^o - \alpha = 90^o}$$

All the angles of each right kite face are right angles hence all 6 congruent right kite faces are square faces. Hence the polyhedron is a hexahedron or cube (figure 8)

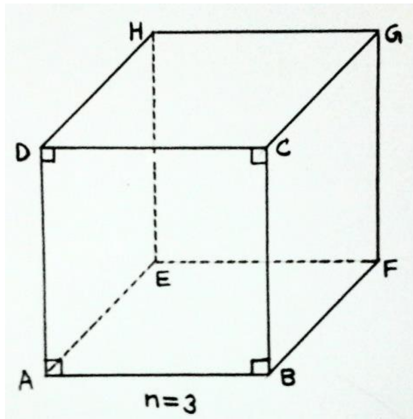

Figure 8: A trapezohedron with 8 vertices, 12 edges & 6 congruent right kite faces is a hexahedron (cube) ABCDEFGH

$$\therefore Surface\ area\ of\ trapezohedron,\ A_s = 2nb^2\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}$$

$$= 2(3)b^2\sqrt{tan\frac{\pi}{3}tan\frac{\pi}{6}} = 6b^2\sqrt{(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}$$

$$= 6b^2\ (surface\ area\ of\ a\ cube\ with\ each\ side\ b)$$

$$\therefore Volume\ of\ trapezohedron,\ V = \frac{nb^3}{3}\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}} = \frac{(3)b^3}{3}\sqrt{tan\frac{\pi}{3}tan\frac{\pi}{6}} = b^3\sqrt{(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}$$

$$= b^3\ \ (volume\ of\ a\ cube\ with\ each\ side\ b)$$

Hence, the **solid angle subtended by n-gonal trapezohedron at each of its two identical & diagonally opposite vertices** is given as follows

$$\boldsymbol{\omega} = 2\pi - 2n\sin^{-1}\left(\sqrt{sin\frac{\pi}{n}tan\frac{\pi}{2n}}\right) = 2\pi - 2(3)\sin^{-1}\left(\sqrt{sin\frac{\pi}{3}tan\frac{\pi}{6}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\sqrt{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{3}}\right)}\right) = 2\pi - 6\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = 2\pi - 6\left(\frac{\pi}{4}\right) = \boldsymbol{\frac{\pi}{2}\ sr}$$

Hence, the **solid angle subtended by n-gonal trapezohedron at each of its 2n identical vertices** is given as follows

$$\omega = 4\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}-\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}}{\sqrt{2}\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)}\right) + 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{n}tan\frac{\pi}{2n}}\sqrt{tan\frac{\pi}{n}tan\frac{\pi}{2n}}-tan\frac{\pi}{n}tan\frac{\pi}{2n}}{\left(1+tan\frac{\pi}{n}tan\frac{\pi}{2n}\right)^{\frac{3}{2}}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{3}tan\frac{\pi}{6}}-\sqrt{tan\frac{\pi}{3}tan\frac{\pi}{6}}}{\sqrt{2}\left(1+tan\frac{\pi}{3}tan\frac{\pi}{6}\right)}\right) + 2\sin^{-1}\left(\frac{\sqrt{1+2tan\frac{\pi}{3}tan\frac{\pi}{6}}\sqrt{tan\frac{\pi}{3}tan\frac{\pi}{6}}-tan\frac{\pi}{3}tan\frac{\pi}{6}}{\left(1+tan\frac{\pi}{3}tan\frac{\pi}{6}\right)^{\frac{3}{2}}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sqrt{1+2(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}-\sqrt{(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}}{\sqrt{2}\left(1+(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)\right)}\right) + 2\sin^{-1}\left(\frac{\sqrt{1+2(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}\sqrt{(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}-(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)}{\left(1+(\sqrt{3})\left(\frac{1}{\sqrt{3}}\right)\right)^{\frac{3}{2}}}\right)$$

$$= 4\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+2\sin^{-1}\left(\frac{\sqrt{3}-1}{(2)^{\frac{3}{2}}}\right) = 4\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)+2\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$= 6\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = \boldsymbol{\frac{\pi}{2}\ sr} \ \ (\textbf{solid angle subtended by a cube at its each vertex})$$

Hence, the dihedral angle between any two adjacent right kite faces meeting at larger common edge (b) is given as

$$2\theta = 2\tan^{-1}\left(\sqrt{cot\frac{\pi}{n}cot\frac{\pi}{2n}}\right) = 2\tan^{-1}\left(\sqrt{cot\frac{\pi}{3}cot\frac{\pi}{6}}\right) = 2\tan^{-1}\left(\sqrt{\left(\frac{1}{\sqrt{3}}\right)(\sqrt{3})}\right) = 2\tan^{-1}(1)$$

$$= 90^{o}\ (angle\ between\ any\ two\ square\ faces\ of\ a\ hexahedron\ or\ cube)$$

Hence, from all above results obtained by the generalised formula it's shown that the n-gonal trapezohedron with 6 congruent right kite faces is a hexahedron or cube. Thus the generalised formula are verified which are equally applicable on any n-gonal trapezohedron with congruent right kite faces. Thus it is clear that there are infinite no. of n-gonal trapezohedron having congruent right kite faces which can be analysed by setting $n = 3, 4, 5, 6, 7, \ldots\ldots\ldots\ldots\ldots\ldots$ to analytically compute all the important parameters such as ratio of unequal sides, outer radius, inner radius, mean radius, surface area, volume, solid angles subtended at the vertices, dihedral angles between the adjacent right kite faces etc.

Conclusions: Let there be any n-gonal trapezohedron/deltohedron having $(2n+2)$ vertices, $4n$ edges & $2n$ congruent faces each as a right kite with two pairs of unequal sides (edges) of $\boldsymbol{a}$ & $\boldsymbol{b}$ then all its important parameters are determined as tabulated below

Inner (inscribed) radius $(\boldsymbol{R_i})$	$R_i = \frac{b}{2}$
Outer (circumscribed) radius $(\boldsymbol{R_o})$	$R_o = \frac{b}{2}\sqrt{2 + tan\frac{\pi}{n} tan\frac{\pi}{2n}}$
Mean radius $(\boldsymbol{R_m})$	$R_m = b\left(\frac{n}{4\pi}\sqrt{tan\frac{\pi}{n} tan\frac{\pi}{2n}}\right)^{\frac{1}{3}}$
Surface area $(\boldsymbol{A_s})$	$A_s = 2nb^2\sqrt{tan\frac{\pi}{n} tan\frac{\pi}{2n}}$
Volume $(\boldsymbol{V})$	$V = \frac{nb^3}{3}\sqrt{tan\frac{\pi}{n} tan\frac{\pi}{2n}}$

An n-gonal trapezohedron has two unequal sides (edges) a & b in a ratio given as

$$\frac{a}{b} = \sqrt{tan\frac{\pi}{n} tan\frac{\pi}{2n}}$$

Each right kite face of n-gonal trapezohedron has one acute angle α given as follows

$$\alpha = 2\tan^{-1}\left(\sqrt{tan\frac{\pi}{n} tan\frac{\pi}{2n}}\right)$$

& two right angles & one obtuse angle β $(\forall\ \beta = 180^o - \alpha)$

Note: Above articles had been developed & *illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

M.M.M. University of Technology, Gorakhpur-273010 (UP) India *April, 2015*

Email: rajpootharishchandra@gmail.com

Author's Home Page: https://notionpress.com/author/HarishChandraRajpoot