$\boldsymbol{Mathematical\ analysis\ of\ identical\ circles\ touching\ one\ another\ on\ the\ whole\ spherical\ surface}$

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Introduction: Here, we are to analyse the identical circles, each having a flat radius r, touching one another on a whole spherical surface with a certain radius R by finding out

1. **Relation between R & r**
2. **Radius of each circle as a great circle arc $(arc\ r)$** &
3. **Total surface area & its percentage** $(\%)$ covered by all the circles on the whole spherical surface.

For such cases, each of the identical circles is assumed to be inscribed by each of the congruent regular polygonal (flat) faces of a platonic solid & spherical surface to be concentric with that platonic solid. Thus each of identical circles touches other circles exactly at the mid-points of edges of each face of the platonic solid. Then by joining the mid-point of one of the edges (also the point of tangency) to the centre of platonic solid (also the centre of the spherical surface), a right triangle, having its hypotenuse R and orthogonal sides (i.e. legs) r & h, is thus obtained. Where,

$$h = normal\ distance\ of\ each\ (flat) face\ from\ the\ centre\ of\ the\ platonic\ solid$$

$$= normal\ distance\ of\ each\ plane\ (flat)\ circle\ from\ the\ centre\ of\ the\ spherical\ surface$$

Here, we are not to derive the parameter such as normal distance of each face from the centre of the different platonic solids. We will prefer to directly use the expressions of **normal distance (h) of each face from the centre of platonic solids** from the **"Table for the important parameters of all five platonic solids"** prepared by the author to derive the mathematical relations of R & r in all the cases to be discussed & analysed here. All five cases will be discussed & analysed in the order corresponding to all five platonic solids.

1. Four identical circles, each having a flat radius r, touching one another at 6 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius R : (Based on Regular Tetrahedron):

In this case, let's assume that each of four identical circles, with a flat radius r, is inscribed by each of four congruent equilateral triangular faces of a regular tetrahedron with an edge length a such that regular tetrahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 6 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 6 edges of a regular tetrahedron. Now, consider one of the four identical circles with the centre C on the flat face & a flat radius r, touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of regular tetrahedron) and is inscribed by an equilateral triangular face of regular tetrahedron with an edge length a . (As shown in the figure 1)

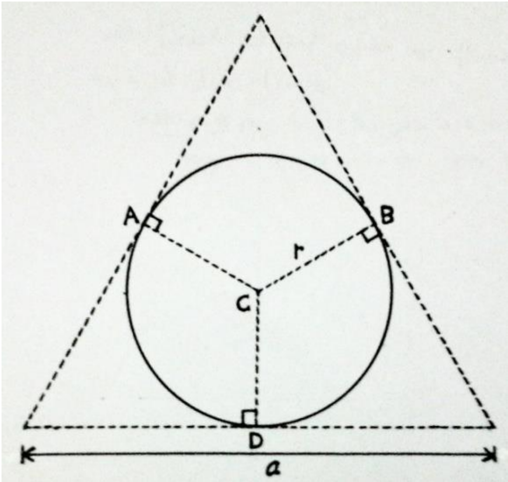

Figure 1: One of 4 identical circles, with the centre C on the flat face & a flat radius r, is touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of face) and is inscribed by an equilateral triangular face of a regular tetrahedron (concentric with the spherical surface) with an edge length a

We know that the **inscribed radius r & the edge length a of a regular n-gon are related** as follows

$$r = \frac{a}{2}cot\frac{\pi}{n} \quad \Rightarrow \ a = 2rtan\frac{\pi}{n}$$

Hence, by setting $n = 3$ for an equilateral triangular face, we get

$$a = 2r\tan\frac{\pi}{3} \quad \Rightarrow \quad \boldsymbol{a = 2r\sqrt{3}}$$

Now, we have

$$h = normal\ distance\ of\ each\ face\ from\ the\ centre\ of\ the\ regular\ tetrahedron\ with\ edge\ length\ a$$

$$= normal\ distance\ of\ plane\ (flat)\ circle\ with\ centre\ C\ from\ the\ centre\ O\ of\ spherical\ surface$$

$$\Rightarrow h = OC = \frac{a}{2\sqrt{6}} \qquad (from\ the\ table\ of\ platonic\ solids)$$

$$= \frac{(2r\sqrt{3})}{2\sqrt{6}} = \frac{r}{\sqrt{2}} \qquad (by\ setting\ the\ value\ of\ edge\ length\ a)$$

$$\Rightarrow \boldsymbol{h = OC = \frac{r}{\sqrt{2}}}$$

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular tetrahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular tetrahedron) to the centre O of the spherical surface (i.e. the centre of regular tetrahedron). Thus, we obtain a right ΔOCA (as shown in the figure 2)

Applying Pythagoras Theorem in right ΔOCA as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = \left(\frac{r}{\sqrt{2}}\right)^2 + r^2 = \frac{3}{2}r^2 \qquad \Rightarrow r^2 = \frac{2}{3}R^2$$

$$\therefore \boldsymbol{flat\ radius\ of\ each\ circle,\ \ r = R\sqrt{\frac{2}{3}} \approx 0.81649658\ R}$$

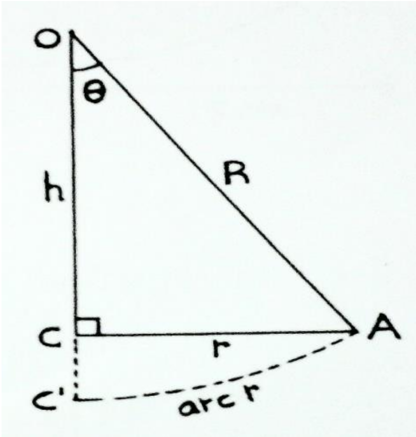

Figure 2: One of 4 identical circles has its centre C on the flat face of tetrahedron & its flat radius r while its centre C' & a radius $arc\ r$ as a great circle arc on the spherical surface

Arc radius ($arc\ r$) of each of 4 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA

$$\sin\theta = \frac{CA}{OA} = \frac{r}{R} = \sqrt{\frac{2}{3}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

$$\Rightarrow \theta = \frac{arc\ AC'}{R} \Rightarrow arc\ radius\ = acr\ AC' = R\theta = R\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$$

$$\therefore \boldsymbol{radius\ of\ each\ circle\ as\ a\ great\ circle\ arc,\ \ arc\ r = R\sin^{-1}\left(\sqrt{\frac{2}{3}}\right) \approx 0.955316618\ R}$$

Total area (A_s) covered by the four identical circles on the spherical surface: In order to calculate the area covered by each of the four identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (See the figure 2 above) by using **formula of solid angle of a right cone concentric with a sphere** as follows

$$Solid\ angle\ subtended\ by\ each\ circle\ at\ the\ centre\ of\ the\ spherical\ surface, \omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow \ \omega = 2\pi\left(1 - \frac{1}{\sqrt{3}}\right) \qquad \left(setting\ the\ value\ of\ \cos\theta = \frac{1}{\sqrt{3}}\ from\ the\ figure\ 2\ above\right)$$

Hence, the **total surface area covered by all 4 identical circles on the sphere**, is given as

$$A_s = (no. of\ circles) \times (solid\ angle\ (\omega)\ subtended\ by\ each\ circle) \times (R^2) = 4\left(2\pi\left(1 - \frac{1}{\sqrt{3}}\right)\right)R^2$$

$$= 8\pi R^2\left(1 - \frac{1}{\sqrt{3}}\right)$$

$$\therefore\ \boldsymbol{A_s = 8\pi R^2\left(1 - \frac{1}{\sqrt{3}}\right) \approx 10.62234631\ R^2}$$

Hence, the **percentage of total surface area covered by all 4 identical circles on the sphere**, is given as

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = \frac{\boldsymbol{total\ surface\ area\ covered\ by\ all\ the\ circles}}{\boldsymbol{total\ surface\ area\ of\ the\ sphere}} \times \boldsymbol{100}$$

$$= \frac{8\pi R^2\left(1 - \frac{1}{\sqrt{3}}\right)}{4\pi R^2} \times 100 = 200\left(1 - \frac{1}{\sqrt{3}}\right)\ \%$$

$$\therefore\ \%\ \boldsymbol{of\ total\ surface\ area\ covered = 200\left(1 - \frac{1}{\sqrt{3}}\right)\ \% \approx 84.53\ \%}$$

KEY POINT-1: 4 identical circles, touching one another at 6 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately $84.53\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $15.47\ \%$ of total surface area is left uncovered by the circles.

2. Six identical circles, each having a flat radius r, touching one another at 12 different points (i.e. each one touches four other circles) on the whole spherical surface with a radius R (Based on Regular Hexahedron i.e. Cube) In this case**,** let's assume that each of 6 identical circles, with a flat radius r, is inscribed by each of 6 congruent square faces of a cube (regular hexahedron) with an edge length a such that cube is concentric with the spherical surface having the centre O & a radius R. In this case, all 12 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 12 edges of a regular hexahedron i.e. cube. Now, consider one of 6 identical circles with the centre C on the flat face & a flat radius r, touching four other circles at the points A, B, D & E (lying on the spherical surface as well as on the edges of the cube) and is inscribed by a square face of the cube with an edge length a . (See the figure 3 below)

We know that the **inscribed radius r & the edge length a of a regular n-gon are related** as follows

$$r = \frac{a}{2}cot\frac{\pi}{n} \quad \Rightarrow\ a = 2rtan\frac{\pi}{n}$$

Hence, by setting $n = 4$ for a square face, we get

$$a = 2rtan\frac{\pi}{4} \quad \Rightarrow\ \boldsymbol{a = 2r}$$

Now, we have

$$h = normal\ distance\ of\ each\ face\ from\ the\ centre\ of\ the\ cube\ with\ edge\ length\ a$$

$$= normal\ distance\ of\ plane\ (flat)\ circle\ with\ centre\ C\ from\ the\ centre\ O\ of\ the\ spherical\ surface$$

$$\Rightarrow\ h = OC = \frac{a}{2} \qquad (from\ the\ table\ of\ platonic\ solids)$$

$$= \frac{(2r)}{2} = r \qquad (by\ setting\ the\ value\ of\ edge\ length\ a)$$

$$\Rightarrow\ \boldsymbol{h = OC = r}$$

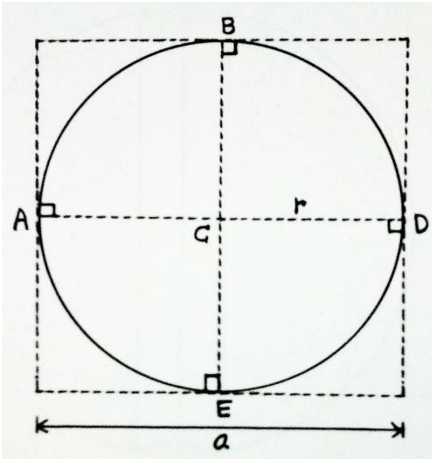

Figure 3: One of 6 identical circles, with the centre C on the flat face & a flat radius r, is touching four other circles at the points A, B, D & E (lying on the spherical surface as well as on the edges of face) and is inscribed by a square face of a cube (concentric with the spherical surface) with an edge length a

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of the cube) to the centre C of the plane (flat) circle & join any of the points A, B, D & E of tangency of the plane circle say point A (i.e. mid-point of one of the edges of cube) to the centre O of the spherical surface (i.e. the centre of cube). Thus, we obtain a right ΔOCA (as shown in the figure 4 below)

Applying Pythagoras Theorem in right ΔOCA as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = (r)^2 + r^2 = 2r^2 \qquad \Rightarrow r^2 = \frac{1}{2}R^2$$

$$\therefore\ \boldsymbol{flat\ radius\ of\ each\ circle,\ r = \frac{R}{\sqrt{2}} \approx 0.707106781\ R}$$

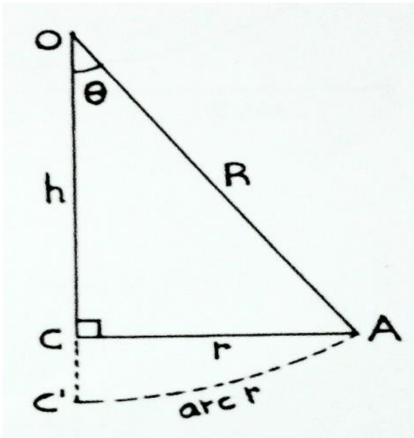

Figure 4: One of 6 identical circles has its centre C on the flat face of a cube & its flat radius r while its centre C' & a radius $arc\ r$ as a great circle arc on the spherical surface

Arc radius ($arc\ r$) of each of 6 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA

$$sin\theta = \frac{CA}{OA} = \frac{r}{R} = \frac{1}{\sqrt{2}} \quad \Rightarrow\ \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

$$\Rightarrow\ \theta = \frac{arc\ AC'}{R} \Rightarrow arc\ radius\ = acr\ AC' = R\theta = \frac{\pi}{4}R$$

$$\therefore\ \boldsymbol{radius\ of\ each\ circle\ as\ a\ great\ circle\ arc,\ \ arc\ r = \frac{\pi R}{4} \approx 0.785398163\ R}$$

Total area (A_s) covered by 6 identical circles on the spherical surface: In order to calculate the area covered by each of the six identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (See the figure 4 above) by using **formula of solid angle of a right cone concentric with a sphere** as follows

$$Solid\ angle\ subtended\ by\ each\ circle\ at\ the\ centre\ of\ the\ spherical\ surface, \omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow\ \omega = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right) \qquad \left(setting\ the\ value\ of\ \cos\theta = \frac{1}{\sqrt{2}}\ \ from\ the\ figure\ 4\ above\right)$$

Hence, the **total surface area covered by all 6 identical circles on the sphere**, is given as

$$A_s = (no.of\ circles) \times (solid\ angle\ (\omega)\ subtended\ by\ each\ circle) \times (R^2) = 6\left(2\pi\left(1-\frac{1}{\sqrt{2}}\right)\right)R^2$$

$$= 12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right)$$

$$\therefore\ \boldsymbol{A_s = 12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right) \approx 11.04181421\ R^2}$$

Hence, the **percentage of total surface area covered by all 6 identical circles on the sphere**, is given as

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = \frac{\boldsymbol{total\ surface\ area\ covered\ by\ all\ the\ circles}}{\boldsymbol{total\ surface\ area\ of\ the\ sphere}} \times \mathbf{100}$$

$$= \frac{12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right)}{4\pi R^2} \times 100 = 300\left(1-\frac{1}{\sqrt{2}}\right)\ \%$$

$$\therefore\ \%\ \boldsymbol{of\ total\ surface\ area\ covered = 300\left(1-\frac{1}{\sqrt{2}}\right)\ \% \approx 87.87\ \%}$$

KEY POINT-2: 6 identical circles, touching one another at 12 different points (i.e. each one touches four other circles) on a whole spherical surface, always cover up approximately $87.87\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $12.13\ \%$ of total surface area is left uncovered by the circles.

3. Eight identical circles, each having a flat radius r, touching one another at 12 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius R (Based on Regular Octahedron): In this case**,** let's assume that each of 8 identical circles, with a flat radius r, is inscribed by each of 8 congruent equilateral triangular faces of a regular octahedron with an edge length a such that regular octahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 12 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 12 edges of a regular octahedron. Now, consider one of 8 identical circles with the centre C on the flat face & a flat radius r, touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of regular octahedron) and is inscribed by an equilateral triangular face of regular octahedron with an edge length a . (As shown in the figure 5)

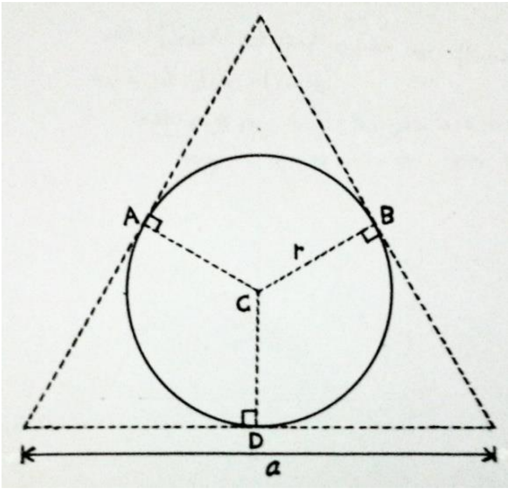

Figure 5: One of 8 identical circles, with the centre C on the flat face & a flat radius r, is touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of face) and is inscribed by an equilateral triangular face of a regular octahedron (concentric with the spherical surface) with an edge length a

We know that the **inscribed radius r & the edge length a of a regular n-gon are related** as follows

$$r = \frac{a}{2}cot\frac{\pi}{n} \quad \Rightarrow\ a = 2rtan\frac{\pi}{n}$$

Hence, by setting $n = 3$ for an equilateral triangular face, we get

$$a = 2rtan\frac{\pi}{3} \quad \Rightarrow\ \boldsymbol{a = 2r\sqrt{3}}$$

Now, we have

$$h = normal\ distance\ of\ each\ face\ from\ the\ centre\ of\ the\ regular\ octahedron\ with\ edge\ length\ a$$

$$= normal\ distance\ of\ plane\ (flat)\ circle\ with\ centre\ C\ from\ the\ centre\ O\ of\ the\ spherical\ surface$$

$$\Rightarrow\ h = OC = \frac{a}{\sqrt{6}} \qquad (from\ the\ table\ of\ platonic\ solids)$$

$$= \frac{(2r\sqrt{3})}{\sqrt{6}} = r\sqrt{2} \qquad (by\ setting\ the\ value\ of\ edge\ length\ a)$$

$$\Rightarrow\ \boldsymbol{h = OC = r\sqrt{2}}$$

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular octahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular octahedron) to the centre O of the spherical surface (i.e. the centre of regular octahedron). Thus, we obtain a right ΔOCA (as shown in the figure 6)

Applying Pythagoras Theorem in right ΔOCA as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = \left(r\sqrt{2}\right)^2 + r^2 = 3r^2 \qquad \Rightarrow r^2 = \frac{1}{3}R^2$$

$$\therefore\ \boldsymbol{flat\ radius\ of\ each\ circle,\ \ r = \frac{R}{\sqrt{3}} \approx 0.577350269\ R}$$

Arc radius ($arc\ r$) of each of 8 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

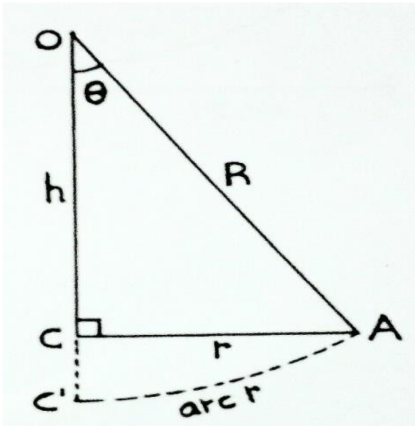

Figure 6: One of 8 identical circles has its centre C on the flat face of octahedron & its flat radius r while its centre C' & a radius $arc\ r$ as a great circle arc on the spherical surface

In right ΔOCA

$$sin\theta = \frac{CA}{OA} = \frac{r}{R} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow\ \theta = \frac{arc\ AC'}{R} \Rightarrow arc\ radius\ = acr\ AC' = R\theta = R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\therefore\ \boldsymbol{radius\ of\ each\ circle\ as\ a\ great\ circle\ arc,\ \ arc\ r = R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.615479708\ R}$$

Total area (A_s) covered by 8 identical circles on the spherical surface: In order to calculate the area covered by each of the eight identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (See the figure 6 above) by using **formula of solid angle of a right cone concentric with a sphere** as follows

$$Solid\ angle\ subtended\ by\ each\ circle\ at\ the\ centre\ of\ the\ spherical\ surface, \omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow\ \omega = 2\pi\left(1 - \sqrt{\frac{2}{3}}\right) \qquad \left(setting\ the\ value\ of\ \cos\theta = \sqrt{\frac{2}{3}}\ \ from\ the\ figure\ 6\ above\right)$$

Hence, the **total surface area covered by all 8 identical circles on the sphere**, is given as

$$A_s = (no.of\ circles) \times (solid\ angle\ (\omega)\ subtended\ by\ each\ circle) \times (R^2) = 8\left(2\pi\left(1-\sqrt{\frac{2}{3}}\right)\right)R^2$$

$$= 16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right)$$

$$\therefore\ \boldsymbol{A_s = 16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right) \approx 9.223887892\ R^2}$$

Hence, the **percentage of total surface area covered by all 8 identical circles on the sphere**, is given as

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = \frac{\boldsymbol{total\ surface\ area\ covered\ by\ all\ the\ circles}}{\boldsymbol{total\ surface\ area\ of\ the\ sphere}} \times \mathbf{100}$$

$$= \frac{16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right)}{4\pi R^2} \times 100 = 400\left(1-\sqrt{\frac{2}{3}}\right)\%$$

$$\therefore\ \%\ \boldsymbol{of\ total\ surface\ area\ covered = 400\left(1-\sqrt{\frac{2}{3}}\right)\% \approx 73.4\ \%}$$

KEY POINT-3: 8 identical circles, touching one another at 12 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately $73.4\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $26.6\ \%$ of total surface area is left uncovered by the circles.

4. Twelve identical circles, each having a flat radius r, touching one another at 30 different points (i.e. each one touches five other circles) on the whole spherical surface with a radius R (Based on Regular dodecahedron): In this case, let's assume that each of 12 identical circles, with a flat radius r, is inscribed by each of 12 congruent regular pentagonal faces of a regular dodecahedron with an edge length a such that regular dodecahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 30 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 30 edges of a regular dodecahedron. Now, consider one of the 12 identical circles with the centre C on the flat face & a flat radius r, touching five other circles at the points A, B, D, E & F (lying on the spherical surface as well as on the edges of the dodecahedron) and is inscribed by a regular pentagonal face of the dodecahedron with an edge length a . (See the figure 7 below)

We know that the **inscribed radius r & the edge length a of a regular n-gon are related** as follows

$$r = \frac{a}{2}\cot\frac{\pi}{n} \quad \Rightarrow a = 2r\tan\frac{\pi}{n}$$

Hence, by setting $n = 4$ for a square face, we get

$$a = 2r\tan\frac{\pi}{5} \quad \Rightarrow \boldsymbol{a = 2r\sqrt{5-2\sqrt{5}}}$$

Now, we have

$$h = normal\ distance\ of\ each\ face\ from\ the\ centre\ of\ the\ dodecahedron\ with\ edge\ length\ a$$

$$= normal\ distance\ of\ plane\ (flat)\ circle\ with\ centre\ C\ from\ the\ centre\ O\ of\ the\ spherical\ surface$$

$$\Rightarrow h = OC = \frac{(3+\sqrt{5})a}{2\sqrt{10-2\sqrt{5}}} \quad (from\ the\ table\ of\ platonic\ solids)$$

$$= \frac{(3+\sqrt{5})\left(2r\sqrt{5-2\sqrt{5}}\right)}{2\sqrt{10-2\sqrt{5}}} \quad (by\ setting\ the\ value\ of\ edge\ length\ a)$$

$$= r\sqrt{\frac{(5-2\sqrt{5})(3+\sqrt{5})^2}{10-2\sqrt{5}}} = r\sqrt{\frac{(5-2\sqrt{5})(7+3\sqrt{5})}{5-\sqrt{5}}} = r\sqrt{\frac{(5+\sqrt{5})(5+\sqrt{5})}{25-5}}$$

$$= \frac{(5+\sqrt{5})r}{2\sqrt{5}} = \frac{(1+\sqrt{5})r}{2}$$

$$\Rightarrow \boldsymbol{h = OC = \frac{(1+\sqrt{5})r}{2}}$$

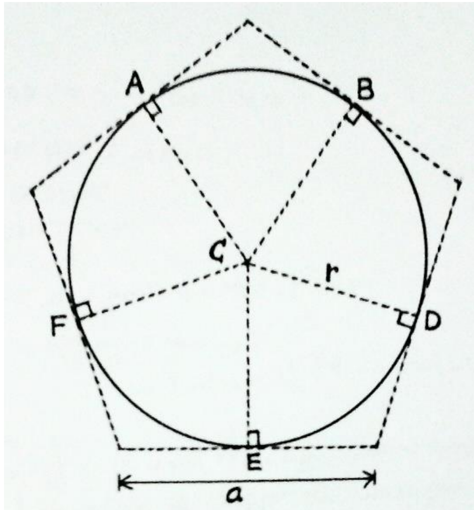

Figure 7: One of 12 identical circles, with the centre C on the flat face & a flat radius r, is touching five other circles at the points A, B, D, E & F (lying on the spherical surface as well as on the edges of face) and is inscribed by a regular pentagonal face of a dodecahedron (concentric with the spherical surface) with an edge length a

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of the regular dodecahedron) to the centre C of the plane (flat) circle & join any of the points A, B, D, E & F of tangency of the plane circle say point A (i.e. mid-point of one of the edges of dodecahedron) to the centre O of the spherical surface (i.e. the centre of dodecahedron). Thus, we obtain a right ΔOCA (as shown in the figure 8 below)

Applying Pythagoras Theorem in right ΔOCA as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \quad \Rightarrow R^2 = \left(\frac{(1+\sqrt{5})r}{2}\right)^2 + r^2 = r^2\left(\frac{6+2\sqrt{5}+4}{4}\right) = r^2\left(\frac{5+\sqrt{5}}{2}\right)$$

$$\Rightarrow r^2 = R^2\left(\frac{2}{5+\sqrt{5}}\right) = R^2\left(\frac{2(5-\sqrt{5})}{(5+\sqrt{5})(5-\sqrt{5})}\right) = R^2\left(\frac{2(5-\sqrt{5})}{20}\right) = R^2\frac{(5-\sqrt{5})}{10}$$

$$\boldsymbol{\therefore\ flat\ radius\ of\ each\ circle,\ r = R\sqrt{\frac{5-\sqrt{5}}{10}} \approx 0.525731112\,R}$$

Arc radius ($arc\ r$) of each of 12 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA

$$sin\theta = \frac{CA}{OA} = \frac{r}{R} = \sqrt{\frac{5-\sqrt{5}}{10}} \quad \Rightarrow \theta = \sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$$

$$\Rightarrow \theta = \frac{arc\ AC'}{R} \Rightarrow arc\ radius\ = acr\ AC' = R\theta = R\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$$

$$\therefore\ \boldsymbol{radius\ of\ each\ circle\ as\ a\ great\ circle\ arc,\ \ arc\ r = R\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)}$$

$$\boldsymbol{\approx 0.553574358\ R}$$

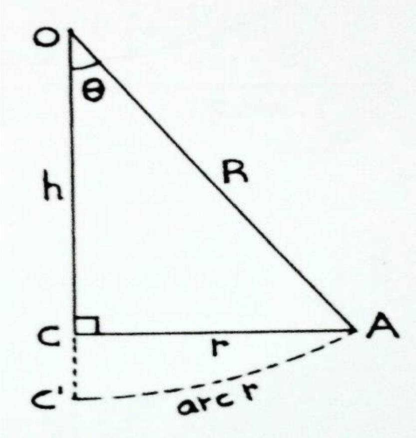

Figure 8: One of 12 identical circles has its centre C on the flat face of a regular dodecahedron & its flat radius r while its centre C' & a radius $arc\ r$ as a great circle arc on the spherical surface

Total area (A_s) covered by 12 identical circles on the spherical surface: In order to calculate the area covered by each of 12 identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (See the figure 8) by using **formula of solid angle of a right cone concentric with a sphere** as follows

$$Solid\ angle\ subtended\ by\ each\ circle\ at\ the\ centre\ of\ the\ spherical\ surface,$$
$$\omega = 2\pi(1-cos\theta)$$

$$\Rightarrow \omega = 2\pi\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right) \quad \left(setting\ the\ value\ of\ \cos\theta = \sqrt{\frac{5+\sqrt{5}}{10}}\ \ from\ fig.8\right)$$

Hence, the **total surface area covered by all 12 identical circles on the sphere**, is given as

$$A_s = (no.of\ circles) \times (solid\ angle\ (\omega)\ subtended\ by\ each\ circle) \times (R^2)$$

$$= 12\left(2\pi\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right)\right)R^2 = 24\pi R^2\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right)$$

$$\therefore\ \boldsymbol{A_s = 24\pi R^2\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right) \approx 11.26066376\ R^2}$$

Hence, the **percentage of total surface area covered by all 12 identical circles on the sphere**, is given as

$$\boldsymbol{\%\ of\ total\ surface\ area\ covered = \frac{total\ surface\ area\ covered\ by\ all\ the\ circles}{total\ surface\ area\ of\ the\ sphere} \times 100}$$

$$= \frac{24\pi R^2\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right)}{4\pi R^2} \times 100 = 600\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right)\%$$

$$\therefore\ \boldsymbol{\%\ of\ total\ surface\ area\ covered = 600\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right)\% \approx 89.61\ \%}$$

KEY POINT-4: 12 identical circles, touching one another at 30 different points (i.e. each one touches five other circles) on a whole spherical surface, always cover up approximately $89.61\ \%$ of total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $10.39\ \%$ of total surface area is left uncovered by the circles.

5. Twenty identical circles, each having a flat radius r, touching one another at 30 different points (i.e. each one touches three other circles) on the whole spherical surface with a radius R (Based on Regular Icosahedron): In this case**,** let's assume that each of 20 identical circles, with a flat radius r, is inscribed by each of 20 congruent equilateral triangular faces of a regular icosahedron with an edge length a such that regular icosahedron is concentric with the spherical surface having the centre O & a radius R. In this case, all 30 points of tangency, lying on the spherical surface, are coincident with the mid-points of all 30 edges of a regular icosahedron. Now, consider one of the 12 identical circles with the centre C on the flat face & a flat radius r, touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of regular icosahedron) and is inscribed by an equilateral triangular face of regular icosahedron with an edge length a . (As shown in the figure 9)

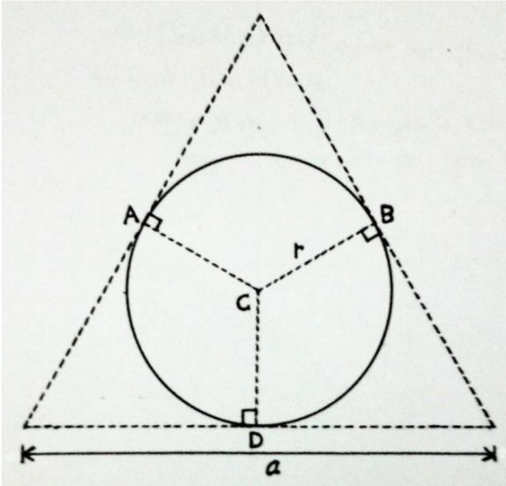

Figure 9: One of 20 identical circles, with the centre C on the flat face & a flat radius r, is touching three other circles at the points A, B & D (lying on the spherical surface as well as on the edges of face) and is inscribed by an equilateral triangular face of a regular icosahedron (concentric with the spherical surface) with an edge length a

We know that the **inscribed radius r & the edge length a of a regular n-gon are related** as follows

$$r = \frac{a}{2}\cot\frac{\pi}{n} \quad \Rightarrow a = 2r\tan\frac{\pi}{n}$$

Hence, by setting $n = 3$ for an equilateral triangular face, we get

$$a = 2r\tan\frac{\pi}{3} \quad \Rightarrow \boldsymbol{a = 2r\sqrt{3}}$$

Now, we have

$$h = normal\ distance\ of\ each\ face\ from\ the\ centre\ of\ the\ regular\ icosahedron\ with\ edge\ length\ a$$

$$= normal\ distance\ of\ plane\ (flat)\ circle\ with\ centre\ C\ from\ the\ centre\ O\ of\ the\ spherical\ surface$$

$$\Rightarrow h = OC = \frac{a(3+\sqrt{5})}{4\sqrt{3}} \qquad (from\ the\ table\ of\ platonic\ solids)$$

$$= \frac{(3+\sqrt{5})(2r\sqrt{3})}{4\sqrt{3}} = \frac{(3+\sqrt{5})r}{2} \qquad (by\ setting\ the\ value\ of\ edge\ length\ a)$$

$$\Rightarrow \boldsymbol{h = OC = \frac{(3+\sqrt{5})r}{2}}$$

Draw the perpendicular OC from the centre O of the spherical surface (i.e. centre of regular icosahedron) to the centre C of the plane (flat) circle & join any of the points A, B & D of tangency of the plane circle say point A (i.e. mid-point of one of the edges of regular icosahedron) to the centre O of the spherical surface (i.e. the centre of regular icosahedron). Thus, we obtain a right ΔOCA (as shown in the figure 10 below)

Applying Pythagoras Theorem in right ΔOCA as follows

$$(OA)^2 = (OC)^2 + (CA)^2 \qquad \Rightarrow R^2 = \left(\frac{(3+\sqrt{5})r}{2}\right)^2 + r^2 = r^2\left(\frac{14+2\sqrt{5}+4}{4}\right) = r^2\left(\frac{9+\sqrt{5}}{2}\right)$$

$$\Rightarrow r^2 = R^2\left(\frac{2}{9+\sqrt{5}}\right) = R^2\left(\frac{2(9-\sqrt{5})}{(9+\sqrt{5})(9-\sqrt{5})}\right) = R^2\left(\frac{2(9-\sqrt{5})}{(81-5)}\right)$$

$$= R^2\left(\frac{9-\sqrt{5}}{38}\right) \Rightarrow r = R\sqrt{\frac{9-\sqrt{5}}{38}}$$

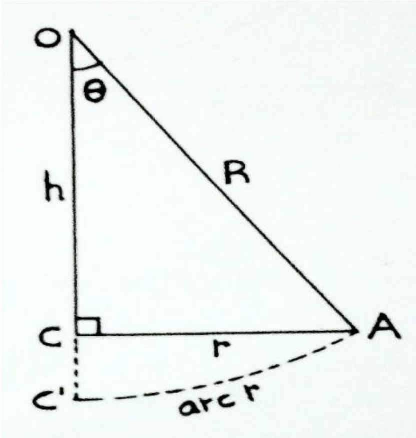

Figure 10: One of 20 identical circles has its centre C on the flat face of icosahedron & its flat radius r while its centre C' & a radius $arc\ r$ as a great circle arc on the spherical surface

$$\therefore\ \boldsymbol{flat\ radius\ of\ each\ circle,\ \ r = R\sqrt{\frac{9-\sqrt{5}}{38}} \approx 0.421898342\ R}$$

Arc radius ($arc\ r$) of each of 20 identical circles: Consider arc radius AC' on the spherical surface with a radius R then we have

In right ΔOCA

$$sin\theta = \frac{CA}{OA} = \frac{r}{R} = \sqrt{\frac{9-\sqrt{5}}{38}} \ \Rightarrow\ \theta = \sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right)$$

$$\Rightarrow\ \theta = \frac{arc\ AC'}{R} \Rightarrow arc\ radius\ = acr\ AC' = R\theta = R\sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right)$$

$$\therefore\ \boldsymbol{radius\ of\ each\ circle\ as\ a\ great\ circle\ arc,\ \ arc\ r = R\sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right) \approx 0.435538116\ R}$$

Total area (A_s) covered by 20 identical circles on the spherical surface: In order to calculate the area covered by each of 20 identical circles on the spherical surface with a radius R, let's first find out the solid angle subtended by each circle with a flat radius r at the centre O of the spherical surface (See the figure 10 above) by using **formula of solid angle of a right cone concentric with a sphere** as follows

$$Solid\ angle\ subtended\ by\ each\ circle\ at\ the\ centre\ of\ the\ spherical\ surface, \omega = 2\pi(1 - cos\theta)$$

$$\Rightarrow\ \omega = 2\pi\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right) \qquad \left(setting\ the\ value\ of\ \cos\theta = \sqrt{\frac{29+\sqrt{5}}{38}}\ \ from\ the\ figure\ 10\ above\right)$$

Hence, the **total surface area covered by all 20 identical circles on the sphere**, is given as

$$A_s = (no.of\ circles) \times (solid\ angle\ (\omega)\ subtended\ by\ each\ circle) \times (R^2)$$

$$= 20\left(2\pi\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right)\right)R^2 = 40\pi R^2\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right)$$

$$\therefore\ \boldsymbol{A_s = 40\pi R^2\left(1 - \sqrt{\frac{29+\sqrt{5}}{38}}\right) \approx 11.73156864\ R^2}$$

Hence, the **percentage of total surface area covered by all 20 identical circles on the sphere**, is given as

$$\%\ \boldsymbol{of\ total\ surface\ area\ covered} = \frac{\boldsymbol{total\ surface\ area\ covered\ by\ all\ the\ circles}}{\boldsymbol{total\ surface\ area\ of\ the\ sphere}} \times \mathbf{100}$$

$$= \frac{40\pi R^2\left(1-\sqrt{\frac{29+\sqrt{5}}{38}}\right)}{4\pi R^2} \times 100 = 1000\left(1-\sqrt{\frac{29+\sqrt{5}}{38}}\right)\%$$

$$\therefore\ \%\ \boldsymbol{of\ total\ surface\ area\ covered} = \mathbf{1000}\left(\mathbf{1}-\sqrt{\frac{\mathbf{29}+\sqrt{\mathbf{5}}}{\mathbf{38}}}\right)\% \approx \mathbf{93.36}\ \%$$

KEY POINT-5: 20 identical circles, touching one another at 30 different points (i.e. each one touches three other circles) on a whole spherical surface, always cover up approximately $93.36\ \%$ of the total surface area irrespective of the radius of the circles or the radius of the spherical surface while approximately $6.64\ \%$ of total surface area is left uncovered by the circles.

Thus, 8 identical circles, touching one another at 12 different points (each one touches three other circles), cover up the minimum approximately $73.4\ \%$ while 20 identical circles, touching one another at 30 different points (each one touches three other circles), cover up the maximum approximately $93.36\ \%$ of the total surface area of a sphere.

Conclusion: Let there a certain no. of the identical circles touching one another on a whole spherical surface with a radius R then we can easily find out the important parameters such as flat radius & arc radius of each circle and surface area covered by the circles on the sphere as tabulated below

Total no. of identical circles touching one another on a whole spherical surface with a radius R	No. of circles touching each circle	Total no. of points of tangency	Flat radius of each circle (i.e. radius of each plane (flat) circle)	Arc radius of each circle (i.e. radius of each circle as a great circle arc on spherical surface)	Total surface area covered by all the circles	Percentage (%) of total surface area covered by all the circles
4	3	6	$R\sqrt{\frac{2}{3}}$ $\approx 0.81649658\ R$	$R\sin^{-1}\left(\sqrt{\frac{2}{3}}\right)$ $\approx 0.955316618\ R$	$8\pi R^2\left(1-\frac{1}{\sqrt{3}}\right)$ $\approx 10.62234631\ R^2$	84.53 %
6	4	12	$\frac{R}{\sqrt{2}}$ $\approx 0.707106781\ R$	$\frac{\pi R}{4}$ $\approx 0.785398163\ R$	$12\pi R^2\left(1-\frac{1}{\sqrt{2}}\right)$ $\approx 11.04181421\ R^2$	87.87 %
8	3	12	$\frac{R}{\sqrt{3}}$ $\approx 0.577350269\ R$	$R\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $\approx 0.615479708\ R$	$16\pi R^2\left(1-\sqrt{\frac{2}{3}}\right)$ $\approx 9.223887892\ R^2$	73.4 %
12	5	30	$R\sqrt{\frac{5-\sqrt{5}}{10}}$ $\approx 0.525731112\ R$	$R\sin^{-1}\left(\sqrt{\frac{5-\sqrt{5}}{10}}\right)$ $\approx 0.553574358\ R$	$24\pi R^2\left(1-\sqrt{\frac{5+\sqrt{5}}{10}}\right)$ $\approx 11.26066376\ R^2$	89.61 %

20	3	30	$R\sqrt{\frac{9-\sqrt{5}}{38}}$ $\approx 0.421898342\ R$	$R\sin^{-1}\left(\sqrt{\frac{9-\sqrt{5}}{38}}\right)$ $\approx 0.435538116\ R$	$40\pi R^2\left(1-\sqrt{\frac{29+\sqrt{5}}{38}}\right)$ $\approx 11.73156864\ R^2$	93.36 %

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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