$\boldsymbol{Mathematical\ analysis\ of\ sphere\ resting\ in\ the\ vertex\ \ of\ regular}$ & $\boldsymbol{uniform\ polyhedrons, filleting\ of\ faces}$ & $\boldsymbol{packing\ of\ spheres\ in\ the\ vertex}$

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Introduction: Here, we are to make the analysis of the sphere resting in the vertex (corner) of regular & uniform polyhedrons such as platonic solids & uniform polyhedrons with right kite faces (analysed by the author) by using generalised expressions of a right pyramid with regular n-gonal base. Here, the regular & the uniform polyhedron are assumed to be hollow shell & large enough to accommodate the given sphere with a certain radius. The sphere, with a radius R, is assumed to be inside & resting in a vertex (corner) at which n no. of edges meet together such that angle between any two consecutive edges is α & resting sphere touches all n no. of faces meeting at that vertex but does not touch any of n no. of edges meeting at that vertex of a given polyhedron. A sphere is best fitted in a vertex first by truncating that vertex & then by filleting all n no. of faces meeting at that vertex. We are also to analyse the packing of the spheres in right pyramids. First of all, let's derive the general expressions of a right pyramid having regular n-gonal base with edge length a.

1. Derivation of normal height H and angles β & γ of the lateral edge & the lateral face with the geometrical axis of a right pyramid with base as a regular polygon

Let there be a right pyramid with base as a regular polygon $A_1A_2A_3 \dots . A_n$ having **n no. of sides each of length a**, **angle between any two consecutive lateral edges α**, **normal height H**, an acute angle $\boldsymbol{\angle OPA_1 = \beta}$ of the geometrical axis PO with any of the lateral edges & an acute angle $\boldsymbol{\angle OPM = \gamma}$ of the geometrical axis PO with any of the lateral faces (as shown in the figure 1)

Now, join all the vertices $A_1, A_2, A_3, \dots . A_n$ of the base to the centre 'O' thus we obtain 'n' no. of congruent isosceles triangles $\Delta A_1OA_2, \Delta A_2OA_3 \dots \dots \dots \Delta A_nOA_1$

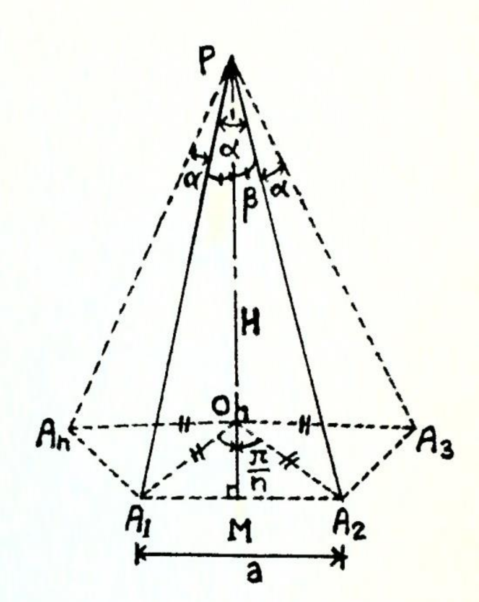

Figure 1: A right pyramid having base as a regular n-gon with each side a, angle between any two consecutive lateral edges $= \alpha$, angle of the geometrical axis PO with each of the lateral edges $= \beta$ & angle of the geometrical axis PO with each of the lateral faces $= \gamma$

In right ΔOMA_2

$$\Rightarrow \tan\angle A_2OM = \frac{MA_2}{OM} \quad or\ \ tan\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{OM}$$

$$\Rightarrow\ \boldsymbol{OM = \frac{a}{2}cot\frac{\pi}{n}} \ \ \left(since, \angle A_1OA_2 = \frac{2\pi}{n}\right) \qquad \dots\dots\dots..(I)$$

Similarly, we have

$$\Rightarrow \sin\angle A_2OM = \frac{MA_2}{OA_2} \quad or\ \ sin\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{OA_2}$$

$$\Rightarrow\ \boldsymbol{OA_2 = \frac{a}{2}cosec\frac{\pi}{n}} \qquad \dots\dots\dots\dots..(II)$$

In right ΔPMA_2

$$\Rightarrow \tan\angle A_2PM = \frac{MA_2}{PM} \quad or\ \ tan\frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{PM}$$

$$\Rightarrow \boldsymbol{PM=\frac{a}{2}cot\frac{\alpha}{2}} \quad \dots\dots\dots\dots\dots\dots.(III)$$

Similarly, we have

$$\Rightarrow \sin\angle A_2PM=\frac{MA_2}{PA_2} \quad or \quad sin\frac{\alpha}{2}=\frac{\left(\frac{a}{2}\right)}{PA_2}$$

$$\Rightarrow \boldsymbol{PA_2=\frac{a}{2}cosec\frac{\alpha}{2}} \quad \dots\dots\dots\dots\dots\dots.(IV)$$

In right $\Delta \boldsymbol{POA_2}$

$$\Rightarrow \sin\angle OPA_2=\frac{OA_2}{PA_2} \quad or \quad \sin\beta=\frac{\left(\frac{a}{2}cosec\frac{\pi}{n}\right)}{\left(\frac{a}{2}cosec\frac{\alpha}{2}\right)}=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}} \quad (from\ eq\ (II)\ \&\ (IV))$$

$$\Rightarrow \boldsymbol{\beta=\sin^{-1}\left(\frac{sin\frac{\alpha}{2}}{sin\frac{\pi}{n}}\right)} \quad \forall\ \boldsymbol{n\geq 3} \quad \dots\dots\dots\dots\dots\dots\dots\dots(V)$$

Above is the generalised formula for calculating the angle $\boldsymbol{\beta}$ between each lateral edge & the geometrical axis of any right pyramid having base as a regular polygon with n no. of sides each of length $\boldsymbol{a}$ & an angle $\boldsymbol{\alpha}$ between any two consecutive lateral edges

In right $\Delta \boldsymbol{POM}$

$$\Rightarrow \sin\angle OPM=\frac{OM}{PM} \quad or \quad \sin\gamma=\frac{\left(\frac{a}{2}cot\frac{\pi}{n}\right)}{\left(\frac{a}{2}cot\frac{\alpha}{2}\right)}=\frac{tan\frac{\alpha}{2}}{tan\frac{\pi}{n}} \quad (from\ eq\ (I)\ \&\ (III))$$

$$\Rightarrow \boldsymbol{\gamma=\sin^{-1}\left(\frac{tan\frac{\alpha}{2}}{tan\frac{\pi}{n}}\right)} \quad \forall\ \boldsymbol{n\geq 3} \quad \dots\dots\dots\dots\dots\dots\dots.(VI)$$

Above is the generalised formula for calculating the angle $\boldsymbol{\gamma}$ between each lateral face & the geometrical axis of any right pyramid having base as a regular polygon with n no. of sides each of length $\boldsymbol{a}$ & an angle $\boldsymbol{\alpha}$ between any two consecutive lateral edges

Similarly, in **right $\Delta \boldsymbol{POM}$,** we have

$$\Rightarrow PM^2=OP^2+OM^2 \quad or \quad PM=\sqrt{H^2+\left(\frac{a}{2}cot\frac{\pi}{n}\right)^2}$$

$$\Rightarrow \boldsymbol{PM=\frac{1}{2}\sqrt{4H^2+a^2cot^2\frac{\pi}{n}}} \quad \dots\dots\dots\dots\dots..(VII)$$

Now, equating the values of **PM** from equation (III) & (VII), we have

$$\Rightarrow \frac{a}{2}cot\frac{\alpha}{2}=\frac{1}{2}\sqrt{4H^2+a^2cot^2\frac{\pi}{n}}$$

On squaring both the sides, we get

$$a^2 cot^2\frac{\alpha}{2} = 4H^2 + a^2 cot^2\frac{\pi}{n} \qquad \Rightarrow 4H^2 = a^2\left(cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}\right)$$

$$\boldsymbol{H = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}} \quad \forall\ \boldsymbol{n \geq 3} \quad \dots\dots\dots\dots\dots\dots\dots\dots.(VIII)$$

Above is the generalised formula for calculating the normal height H of any right pyramid having base as a regular polygon with n no. of sides each of length a & an angle α between any two consecutive lateral edges

2. Locating a sphere with a radius R resting in the vertex (corner) at which n no. of edges meet together at an angle α between any two consecutive of them (Ex. vertex of a regular polyhedron (platonic solid), any of two identical & diagonally opposite vertices of a uniform polyhedron (trapezohedron) with congruent right kite faces and vertex of a right pyramid with regular n-gonal base): Let there be a sphere, having its centre C & a radius R, resting in a vertex (corner) P at which n no. of edges meet together at angle α between any two consecutive of them then this case the vertex can be treated as the vertex of a right pyramid with regular n-gonal base (as shown in the figure (2) below)

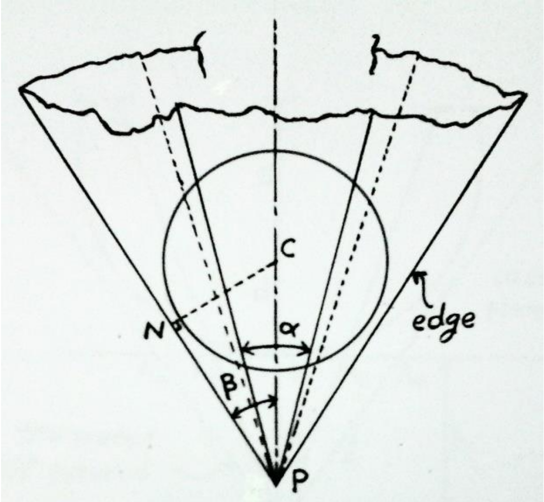

Figure 2: A sphere resting in a vertex (corner) P of a polyhedron touches all n no. of faces but does not touch any of n no. of edges meeting at the vertex P

Now, draw a perpendicular say CQ from the centre C to the any of faces meeting at the vertex P & join the centre C with the vertex P (as shown in the figure (3) below). Let the distance of the centre C from the vertex P be $CP = d$ & all n no. of faces, meeting at the vertex P, are equally inclined at an angle $\angle CPQ = \gamma$ with the geometrical axis PC.

Now, in right ΔPQC (figure 3 below)

$$\sin\angle CPQ = \frac{CQ}{CP} \qquad \Rightarrow \sin\gamma = \frac{R}{d} \quad \Rightarrow \quad d = \frac{R}{\sin\gamma}$$

Now, setting the value of $\sin\gamma$ from eq(VI) as follows

$$d = \frac{R}{\left(\frac{tan\frac{\alpha}{2}}{tan\frac{\pi}{n}}\right)} = \frac{R\,tan\frac{\pi}{n}}{tan\frac{\alpha}{2}}$$

Hence, **the distance ($CP = d$) of the centre of the resting sphere from the vertex of polyhedron** is given as follows

$$\boldsymbol{d = \frac{R\,tan\frac{\pi}{n}}{tan\frac{\alpha}{2}}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n} \quad \&\ \ n \geq 3\right)$$

Hence, **the minimum distance (d_{min}) of the resting sphere from the vertex of polyhedron** is equal to the distance of the point S on the sphere which is closest to the vertex P (as shown in the figure 3). Hence is given as follows

$$PS = PC - CS = d - R = \frac{R\,tan\frac{\pi}{n}}{tan\frac{\alpha}{2}} - R = \frac{R\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)}{tan\frac{\alpha}{2}}$$

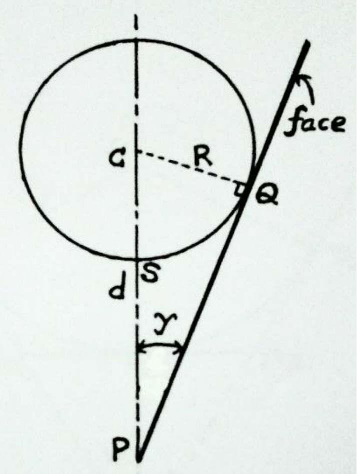

Figure 3: A sphere resting in a vertex (corner) P of a polyhedron is touching the face (shown by the line PQ) normal to the plane of the paper

$$\boldsymbol{d_{min} = \frac{R\left(\tan\frac{\pi}{n} - \tan\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3\right)$$

In right ΔPNC (figure 2 above)

$$\sin\angle CPN = \frac{CN}{CP} \quad \Rightarrow \quad \sin\beta = \frac{CN}{d} \quad \Rightarrow \quad CN = d\sin\beta$$

Now, setting the value of d & the value of $\sin\beta$ from eq(V) in above value of CN as follows

$$CN = \frac{R\tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}}\right) = \frac{R\cos\frac{\alpha}{2}}{\cos\frac{\pi}{n}}$$

Hence, **normal distance (d_e) of the centre of resting sphere from each edge of polyhedron** is given as follows

$$\boldsymbol{d_e = \frac{R\cos\frac{\alpha}{2}}{\cos\frac{\pi}{n}}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3\right)$$

Hence, **the minimum distance ($(d_e)_{min}$) of the resting sphere from each edge of polyhedron** is equal to the distance of a point on the sphere which is closest to the edge (see figure 2 above). Hence it is given as follows

$$(d_e)_{min} = CN - R = d_e - R = \frac{R\cos\frac{\alpha}{2}}{\cos\frac{\pi}{n}} - R = \frac{R\left(\cos\frac{\alpha}{2} - \cos\frac{\pi}{n}\right)}{\cos\frac{\pi}{n}}$$

$$\boldsymbol{(d_e)_{min} = \frac{R\left(\cos\frac{\alpha}{2} - \cos\frac{\pi}{n}\right)}{\cos\frac{\pi}{n}}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3\right)$$

3. Truncation of the vertex (corner) of the polyhedron to best fit the sphere in that vertex: In order to fillet all n no. of faces meeting at the vertex of a polyhedron to best fit a sphere in it. First of all, the vertex is truncated with a cutting plane through **an appropriate normal height** (**depth**) $\boldsymbol{h}$ & then **each of $\boldsymbol{n}$ no. of truncated faces**, initially meeting at the vertex P, is provided an **appropriate fillet radius $\boldsymbol{R_f}$** (as shown in the figure 4).

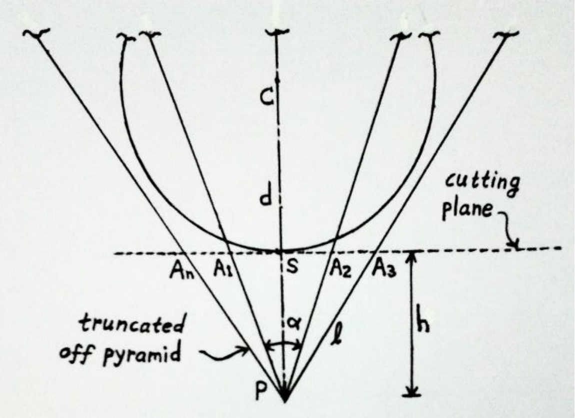

Figure 4: Vertex (corner) P is truncated by a cutting plane (normal to the plane of paper shown by the dotted line) just touching the sphere at the point S

Truncation of the vertex through a normal height (depth): Let the vertex P be truncated with a plane (as shown by the dotted line normal to the plane of paper) through a normal height (depth) h such that the cutting plane just touches the resting sphere, having centre C & a radius R, at the point S. Thus we obtain a truncated off right pyramid with n-gonal base $A_1A_2A_3 \dots . A_n$ having **$\boldsymbol{n}$ no. of sides, angle between any two consecutive lateral edges $\boldsymbol{\alpha}$ & normal height $\boldsymbol{h}$**. Then normal height (depth) h to be cut is given as follows

$$h = PS = minimum\ distance\ of\ sphere\ from\ the\ vertex\ P = d_{min}$$

Hence, the **normal height (depth) $\boldsymbol{h}$, through which the vertex is to be truncated,** is given as

$$\boldsymbol{h = d_{min} = \frac{R\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)}{tan\frac{\alpha}{2}}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3\right)$$

The above formula shows that the vertex, at which n no. of edges meet together such that angle between any two consecutive edges is α, should be truncated by a normal height (depth) h to best fit a sphere with a radius R in that vertex (corner) (of a polyhedron).

Truncation of the vertex through an edge length: Alternatively, the vertex P can also be truncated by cutting each of n no. of edges, meeting at that vertex, through an edge length say l measured from the vertex P (See figure 4 above). Thus we obtain a truncated off right pyramid with n-gonal base $A_1A_2A_3 \dots .A_n$ having $\boldsymbol{n}$ **no. of sides** say **each of length a**, **angle between any two consecutive lateral edges** $\boldsymbol{\alpha}$ **& normal height** $\boldsymbol{h}$. The normal height (depth) h of right pyramid is given from the generalised eq(VIII) as follows

$$h = \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}$$

But $h = d_{min}$, thus equating both the results as follows

$$h = d_{min} \quad \Rightarrow \quad \frac{a}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}} = \frac{R\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)}{tan\frac{\alpha}{2}}$$

$$\Rightarrow \quad a = \frac{2R\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)}{tan\frac{\alpha}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}} = \frac{2R\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)}{tan\frac{\alpha}{2}\sqrt{cot^2\frac{\alpha}{2} - cot^2\frac{\pi}{n}}}$$

$$= \frac{2R\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)tan\frac{\pi}{n}tan\frac{\alpha}{2}}{tan\frac{\alpha}{2}\sqrt{tan^2\frac{\pi}{n} - tan^2\frac{\alpha}{2}}} = \frac{2R\ tan\frac{\pi}{n}\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)}{\sqrt{\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)\left(tan\frac{\pi}{n} + tan\frac{\alpha}{2}\right)}} = 2R\ tan\frac{\pi}{n}\sqrt{\frac{tan\frac{\pi}{n} - tan\frac{\alpha}{2}}{tan\frac{\pi}{n} + tan\frac{\alpha}{2}}}$$

$$= 2R\ tan\frac{\pi}{n}\sqrt{\frac{tan\frac{\pi}{n} - tan\frac{\alpha}{2}}{tan\frac{\pi}{n} + tan\frac{\alpha}{2}}} = 2R\ tan\frac{\pi}{n}\sqrt{\frac{sin\frac{\pi}{n}cos\frac{\alpha}{2} - cos\frac{\pi}{n}sin\frac{\alpha}{2}}{sin\frac{\pi}{n}cos\frac{\alpha}{2} + cos\frac{\pi}{n}sin\frac{\alpha}{2}}} = 2R\ tan\frac{\pi}{n}\sqrt{\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}}$$

$$\Rightarrow \quad \boldsymbol{a = 2R\ tan\frac{\pi}{n}\sqrt{\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}} = A_1A_2} \qquad \dots\dots\dots\dots\dots\dots\dots .(IX)$$

The length (l) of each of n no. of lateral edges of the truncated off right pyramid is given from generalised eq(IV) derived above as follows

$$l = PA_1 = PA_2 = \frac{a}{2}cosec\frac{\alpha}{2} \quad \Rightarrow \quad l = \left(R\ tan\frac{\pi}{n}\sqrt{\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}}\right)cosec\frac{\alpha}{2}$$

Hence, the **edge length** $\boldsymbol{l}$**, through which the vertex is to be truncated,** is given as

$$\boldsymbol{l = R\ tan\frac{\pi}{n}\ cosec\frac{\alpha}{2}\sqrt{\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3\right)$$

The above formula shows that all n no. of edges, meeting together at a vertex such that angle between any two consecutive edges is α, should be truncated by a length l to best fit a sphere with a radius R in that vertex (corner) (of a polyhedron).

4. Filleting of the faces meeting at the vertex & touching the sphere: In order to best fit a sphere in the truncated vertex (corner) P (as discussed in the previous article 3) each of n no. of truncated faces, having apex angle α & meeting together at the vertex P, should be provided a fillet radius R_f with the centre O (as shown in the figure 5). Draw the perpendiculars OT_1 & OT_2 from the points T_1 & T_2 to the truncated edges which upon extending meet at the truncated vertex P at an angle α. Join the centre O, lying on the bisector of $\angle A_1PA_2 = \alpha$, to the vertex P. Thus we have from eq(IX)

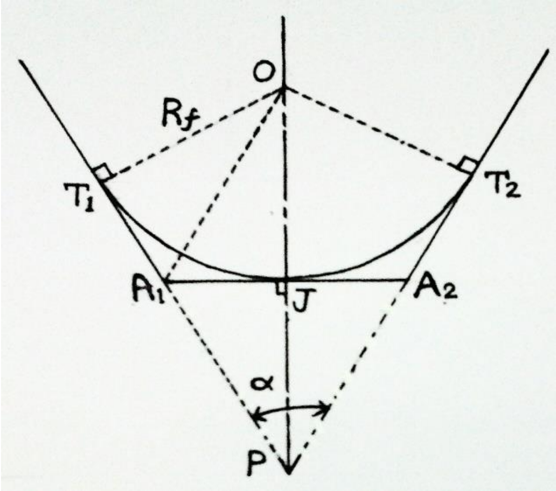

Figure 5: A face, truncated at the edge A_1A_2, is filleted though a radius R_f to best fit a sphere with a radius R in the vertex (corner) (of a polyhedron)

$$\boldsymbol{A_1J = A_2J = \frac{A_1A_2}{2} = \frac{a}{2} = R\tan\frac{\pi}{n}\sqrt{\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}}}$$

$$\angle A_1OJ = \frac{\angle T_1OP}{2} = \frac{\frac{\pi}{2}-\angle OPT_1}{2} = \frac{\frac{\pi}{2}-\frac{\alpha}{2}}{2} = \frac{\pi-\alpha}{4}$$

$$\Rightarrow \boldsymbol{\angle JA_1O} = \frac{\pi}{2} - \angle A_1OJ = \frac{\pi}{2} - \left(\frac{\pi-\alpha}{4}\right) = \boldsymbol{\frac{\pi+\alpha}{4}}$$

In right $\Delta OJ\,A_1$

$$\tan\angle JA_1O = \frac{OJ}{A_1J} \quad \Rightarrow \tan\left(\frac{\pi+\alpha}{4}\right) = \frac{R_f}{A_1J}$$

$$\Rightarrow R_f = (A_1J)\tan\left(\frac{\pi+\alpha}{4}\right) = \left(R\,tan\frac{\pi}{n}\sqrt{\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}}\right)\tan\left(\frac{\pi+\alpha}{4}\right)$$

Hence, the **fillet radius R_f of each truncated face, to best fit the sphere**, is given as

$$\boldsymbol{R_f = R\tan\frac{\pi}{n}\tan\left(\frac{\pi+\alpha}{4}\right)\sqrt{\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n} \ \ \& \ n \geq 3\right)$$

The above formula shows that all n no. of truncated faces with apex angle α, meeting together at a vertex, should be filleted/rounded through a radius R_f to best fit a sphere with a radius R in that vertex (corner) (of a polyhedron).

Illustrative examples on locating the sphere resting in the vertex

1. Regular Tetrahedron:. Consider a sphere with a radius R is resting in one of 4 identical vertices (corners) of a regular tetrahedron. We know that three edges meet at each vertex of a regular tetrahedron at an angle 60^o between any two consecutive of them. Then in this case we have

$$n = 3 \ \& \ \alpha = 60^o = \frac{\pi}{3}$$

Thus we can calculate all the important parameters for a sphere resting in the vertex (corner) of a regular tetrahedron as follows

1. **Distance (d) of the centre of the resting sphere from the vertex of tetrahedron** is given as follows

$$d = \frac{R\tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} = \frac{R\tan\frac{\pi}{3}}{\tan\frac{\pi}{6}} = R\frac{\sqrt{3}}{\left(\frac{1}{\sqrt{3}}\right)} = 3R \qquad \Rightarrow \boxed{\boldsymbol{d = 3R}}$$

2. **Minimum distance (d_{min}) of the resting sphere from the vertex of tetrahedron** is given as follows

$$d_{min} = \frac{R\left(\tan\frac{\pi}{n} - \tan\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}} = \frac{R\left(\tan\frac{\pi}{3} - \tan\frac{\pi}{6}\right)}{\tan\frac{\pi}{6}} = \frac{R\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)}{\left(\frac{1}{\sqrt{3}}\right)} = R(3-1) = 2R$$

$$\Rightarrow \boxed{\boldsymbol{d_{min} = 2R}}$$

3. **Distance (d_e) of the centre of the resting sphere from each edge meeting at the vertex of the tetrahedron** is given as follows

$$d_e = \frac{R\cos\frac{\alpha}{2}}{\cos\frac{\pi}{n}} = \frac{R\cos\frac{\pi}{6}}{\cos\frac{\pi}{3}} = \frac{R\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = R\sqrt{3}$$

$$\Rightarrow \boxed{\boldsymbol{d = R\sqrt{3} \approx 1.732050808\,R}}$$

4. **Minimum distance ($(d_e)_{min}$) of the resting sphere from each edge meeting at the vertex of the tetrahedron** is given as

$$(d_e)_{min} = \frac{R\left(\cos\frac{\alpha}{2} - \cos\frac{\pi}{n}\right)}{\cos\frac{\pi}{n}} = \frac{R\left(\cos\frac{\pi}{6} - \cos\frac{\pi}{3}\right)}{\cos\frac{\pi}{3}} = \frac{R\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = R\left(\sqrt{3} - 1\right)$$

$$\Rightarrow \boxed{\boldsymbol{d = R\left(\sqrt{3} - 1\right) \approx 0.732050807\,R}}$$

5. **Normal height (depth) (h) through which the vertex of the tetrahedron is truncated to best fit the sphere** is given as

$$h = d_{min} = 2R$$

$$\Rightarrow \boxed{\boldsymbol{h = 2R}}$$

6. **Edge length (l) through which the vertex of the tetrahedron is truncated to best fit the sphere** is given as

$$l = R\,tan\frac{\pi}{n}\,cosec\frac{\alpha}{2}\sqrt{\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}} = R\,tan\frac{\pi}{3}\,cosec\frac{\pi}{6}\sqrt{\frac{\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)}} = R(\sqrt{3})(2)\sqrt{\frac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{2}\right)}}$$

$$= 2R\sqrt{3}\sqrt{\frac{\left(\frac{1}{2}\right)}{1}} = R\sqrt{6}$$

$$\Rightarrow \boxed{\boldsymbol{l = R\sqrt{6} \approx 2.449489743\,R}}$$

7. **Fillet radius (R_f) of each truncated face to best fit the sphere in the truncated vertex of the tetrahedron** is given as

$$R_f = R\,tan\frac{\pi}{n}\,tan\left(\frac{\pi+\alpha}{4}\right)\sqrt{\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}} = R\,tan\frac{\pi}{3}\,tan\left(\frac{\pi+\frac{\pi}{3}}{4}\right)\sqrt{\frac{\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)}}$$

$$= R(\sqrt{3})\left(\tan\frac{\pi}{3}\right)\sqrt{\frac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{2}\right)}} = R(\sqrt{3})(\sqrt{3})\sqrt{\frac{1}{2}} = \frac{3R}{\sqrt{2}}$$

$$\Rightarrow \boxed{\boldsymbol{R_f = \frac{3R}{\sqrt{2}} \approx 2.121320344\,R}}$$

The above value is very important to fillet all three faces meeting at a vertex (corner) for best fitting any sphere with a radius R in that vertex of a regular tetrahedron.

2. Regular Hexahedron (cube): Consider a sphere with a radius R is resting in one of 8 identical vertices (corners) of a cube. We know that three edges meet at each vertex of a cube at an angle 90^o between any two consecutive of them. Then in this case we have

$$n = 3\ \ \&\ \alpha = 90^o = \frac{\pi}{2}$$

Thus we can calculate all the important parameters for a sphere resting in the vertex (corner) of a regular hexahedron (cube) as follows

1. **Distance (d) of the centre of the resting sphere from the vertex of cube** is given as follows

$$d = \frac{R\,tan\frac{\pi}{n}}{tan\frac{\alpha}{2}} = \frac{R\,tan\frac{\pi}{3}}{tan\frac{\pi}{4}} = \frac{R\sqrt{3}}{1} = R\sqrt{3} \qquad \Rightarrow \boxed{\boldsymbol{d = R\sqrt{3} \approx 1.732050808\,R}}$$

2. **Minimum distance (d_{min}) of the resting sphere from the vertex of cube** is given as follows

$$d_{min} = \frac{R\left(tan\frac{\pi}{n} - tan\frac{\alpha}{2}\right)}{tan\frac{\alpha}{2}} = \frac{R\left(tan\frac{\pi}{3} - tan\frac{\pi}{4}\right)}{tan\frac{\pi}{4}} = \frac{R(\sqrt{3}-1)}{1} = R(\sqrt{3}-1)$$

$$\Rightarrow \boxed{\boldsymbol{d_{min} = R(\sqrt{3}-1) \approx 0.732050807\,R}}$$

3. **Distance (d_e) of the centre of the resting sphere from each edge meeting at the vertex of the cube** is given as follows

$$d_e = \frac{R\ cos\frac{\alpha}{2}}{cos\frac{\pi}{n}} = \frac{R\ cos\frac{\pi}{4}}{cos\frac{\pi}{3}} = \frac{R\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{2}\right)} = R\sqrt{2}$$

$$\Rightarrow \boxed{\boldsymbol{d = R\sqrt{2} \approx 1.414213562\ R}}$$

4. **Minimum distance ($(d_e)_{min}$) of the resting sphere from each edge meeting at the vertex of the cube** is given as

$$(d_e)_{min} = \frac{R\left(cos\frac{\alpha}{2} - cos\frac{\pi}{n}\right)}{cos\frac{\pi}{n}} = \frac{R\left(cos\frac{\pi}{4} - cos\frac{\pi}{3}\right)}{cos\frac{\pi}{3}} = \frac{R\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)}{\left(\frac{1}{2}\right)} = R\left(\sqrt{2} - 1\right)$$

$$\Rightarrow \boxed{\boldsymbol{d = R\left(\sqrt{2} - 1\right) \approx 0.414213562\ R}}$$

5. **Normal height (depth) (h) through which the vertex of the cube is truncated to best fit the sphere** is given as

$$h = d_{min} = R\left(\sqrt{3} - 1\right)$$

$$\Rightarrow \boxed{\boldsymbol{h = R\left(\sqrt{3} - 1\right) \approx 0.732050807\ R}}$$

6. **Edge length (l) through which the vertex of the cube is truncated to best fit the sphere** is given as

$$l = R\ tan\frac{\pi}{n}\ cosec\frac{\alpha}{2}\sqrt{\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}} = R\ tan\frac{\pi}{3}\ cosec\frac{\pi}{4}\sqrt{\frac{\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)}} = R\left(\sqrt{3}\right)\left(\sqrt{2}\right)\sqrt{\frac{\sin\left(\frac{\pi}{12}\right)}{\sin\left(\frac{7\pi}{12}\right)}}$$

$$= R\sqrt{6}\sqrt{\frac{\sin\left(\frac{\pi}{12}\right)}{\cos\left(\frac{\pi}{12}\right)}} = R\sqrt{6\tan\left(\frac{\pi}{12}\right)} = R\sqrt{6\left(2 - \sqrt{3}\right)} = R\sqrt{\left(3 - \sqrt{3}\right)^2} = R\left(3 - \sqrt{3}\right)$$

$$\Rightarrow \boxed{\boldsymbol{l = R\left(3 - \sqrt{3}\right) \approx 1.267949192\ R}}$$

7. **Fillet radius (R_f) of each truncated face to best fit the sphere in the truncated vertex of the cube** is given as

$$R_f = R\ tan\frac{\pi}{n} tan\left(\frac{\pi + \alpha}{4}\right)\sqrt{\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}} = R\ tan\frac{\pi}{3} tan\left(\frac{\pi + \frac{\pi}{2}}{4}\right)\sqrt{\frac{\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)}}$$

$$= R\left(\sqrt{3}\right)\left(\cot\frac{\pi}{8}\right)\sqrt{\tan\left(\frac{\pi}{12}\right)} = R\left(\sqrt{3}\right)\left(1 + \sqrt{2}\right)\sqrt{2 - \sqrt{3}} = R\left(\sqrt{3} + \sqrt{6}\right)\sqrt{\frac{\left(\sqrt{3} - 1\right)^2}{2}}$$

$$= \frac{R\left(\sqrt{3} + \sqrt{6}\right)\left(\sqrt{3} - 1\right)}{\sqrt{2}} = \frac{R\sqrt{2}\left(3 + 3\sqrt{2} - \sqrt{3} - \sqrt{6}\right)}{2} = \frac{R\left(6 + 3\sqrt{2} - 2\sqrt{3} - \sqrt{6}\right)}{2}$$

$$\Rightarrow \quad \boxed{\boldsymbol{R_f = \frac{R(6+3\sqrt{2}-2\sqrt{3}-\sqrt{6})}{2} \approx 2.164524665\,R}}$$

The above value is very important to fillet all three faces meeting at a vertex (corner) for best fitting any sphere with a radius $\boldsymbol{R}$ in that vertex of a cube.

Similarly, we can derive important expressions for other platonic solids, uniform polyhedrons & right pyramids.

5. Packing of the spheres in the right pyramid with a regular polygonal base: Let there be a right pyramid with base as a regular polygon $A_1A_2A_3 \dots . A_n$ having **$\boldsymbol{n}$ no. of sides each of equal length $\boldsymbol{a}$, angle between any two consecutive lateral edges $\boldsymbol{\alpha}$ & normal height H** (As shown in the figure 6)

Now, consider a largest sphere, having centre C_1 & a radius $R_1(=R)$, completely inscribed in right pyramid such that it touches the polygonal base $A_1A_2A_3 \dots . A_n$ at the centre O as well as all the lateral faces. Further locate a sphere, having centre C_2 & a radius R_2, completely inscribed in the pyramid such that it touches the largest sphere at the point O_1 , the polygonal base $A_1A_2A_3 \dots . A_n$ at the centre O & all the lateral faces. Thus continue to pack the right pyramid by locating smaller & smaller spheres in the same fashion up to total N no. of spheres including the largest one. Thus **total $\boldsymbol{N}$ no. of the spheres,** having **radii $\boldsymbol{R_1, R_2, R_3, \dots \dots \dots . R_N}$ respectively, are snugly fitted touching one another** between the polygonal base & the apex (vertex) point P of the right pyramid.

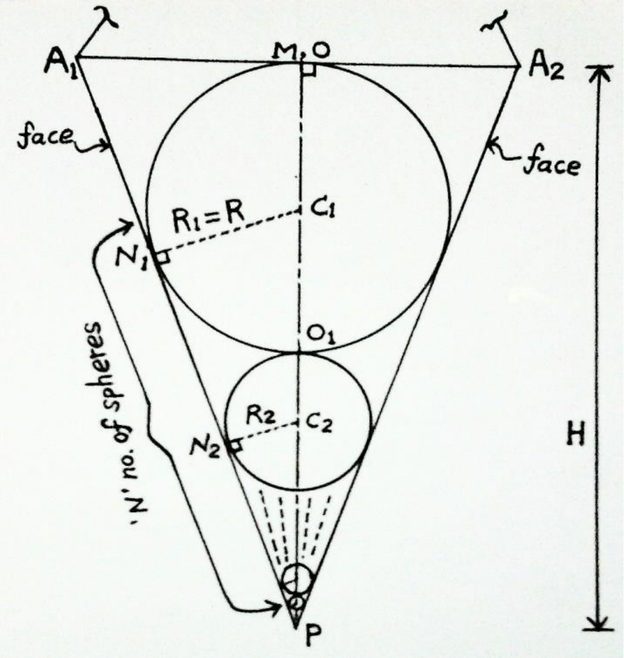

Figure 6: N no. of the spheres are snugly fitted touching one another between the regular polygonal base & the apex point P of a right pyramid. Isosceles triangular faces $\boldsymbol{PA_1}$ & $\boldsymbol{PA_2}$ are inclined to the plane of paper & the face $\boldsymbol{A_1PA_2}$ is in the plane of paper $(\angle \boldsymbol{A_1PA_2 = \alpha})$

Now, the distance of the centre C_1 of the largest (resting) sphere, with a radius $R_1 = R$, from the vertex P of the right pyramid is given by the generalised formula as follows

$$PC_1 = \frac{R_1 \tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} = \frac{R \tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} \qquad (\angle \boldsymbol{A_1PA_2 = \alpha})$$

Hence, the **normal height $\boldsymbol{H}$ of right pyramid** is given as

$$H = PO = PC_1 + C_1O = \frac{R \tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} + R$$

$$\Rightarrow \quad \boldsymbol{H = \frac{R\left(\tan\frac{\pi}{n} + \tan\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}}}$$

But the normal height of the right pyramid is given by generalised formula from eq(VIII) above as follows

$$\boldsymbol{H = \frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}}} \qquad (\,a\ is\ length\ of\ each\ side\ of\ polygonal\ base\ of\ pyramid)$$

Equating both the above values of H, we get

$$\frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2}-\cot^2\frac{\pi}{n}}=\frac{R\left(\tan\frac{\pi}{n}+\tan\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}}\quad\Rightarrow\ a=\frac{2R\left(\tan\frac{\pi}{n}+\tan\frac{\alpha}{2}\right)\tan\frac{\pi}{n}\tan\frac{\alpha}{2}}{\tan\frac{\alpha}{2}\sqrt{\tan^2\frac{\pi}{n}-\tan^2\frac{\alpha}{2}}}$$

$$\Rightarrow\ a=2R\tan\frac{\pi}{n}\sqrt{\frac{\tan\frac{\pi}{n}+\tan\frac{\alpha}{2}}{\tan\frac{\pi}{n}-\tan\frac{\alpha}{2}}}=2R\tan\frac{\pi}{n}\sqrt{\frac{\sin\frac{\pi}{n}\cos\frac{\alpha}{2}+\cos\frac{\pi}{n}\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}\cos\frac{\alpha}{2}-\cos\frac{\pi}{n}\sin\frac{\alpha}{2}}}$$

$$\Rightarrow\ \boldsymbol{a=2R\tan\frac{\pi}{n}\sqrt{\frac{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}}=A_1A_2}$$

Now, the **volume (V_Δ) of the right pyramid** is given as

$$V_\Delta=\frac{1}{3}\times(area\ of\ regular\ polygonal\ base)\times(normal\ height)=\frac{1}{3}\times\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)\times(H)$$

Now, setting the values of length a & normal height H in term of radius R of the largest sphere inscribed in the pyramid, we get the volume of pyramid as follows

$$V_\Delta=\frac{1}{12}n\cot\frac{\pi}{n}\left(2R\tan\frac{\pi}{n}\sqrt{\frac{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}}\right)^2\times\left(\frac{R\left(\tan\frac{\pi}{n}+\tan\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}}\right)$$

$$=\frac{4}{12}nR^3\cot\frac{\pi}{n}\left(\tan^2\frac{\pi}{n}\frac{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}\right)\left(\frac{\sin\frac{\pi}{n}\cos\frac{\alpha}{2}+\cos\frac{\pi}{n}\sin\frac{\alpha}{2}}{\tan\frac{\alpha}{2}\cos\frac{\pi}{n}\cos\frac{\alpha}{2}}\right)$$

$$=\frac{1}{3}nR^3\tan\frac{\pi}{n}\left(\frac{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}\right)\left(\frac{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}\cos\frac{\pi}{n}}\right)=\frac{1}{3}nR^3\left(\frac{\sin\frac{\pi}{n}\sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}\cos^2\frac{\pi}{n}\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}\right)$$

Hence, the **volume (V_Δ) of the right pyramid** is given as

$$\boldsymbol{V_\Delta=\frac{1}{3}nR^3\left(\frac{\sin\frac{\pi}{n}\sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}\cos^2\frac{\pi}{n}\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}\right)}\qquad\left(\forall\ \alpha<\frac{2\pi}{n}\ \ \&\ n\geq3\right)$$

Now, the distance of the centre C_2 of larger (resting) sphere, with a radius R_2, from the vertex P of the right pyramid is given by the generalised formula as follows

$$\boldsymbol{PC_2=\frac{R_2\tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}}}$$

Since the spheres with centres C_1 & C_2 are externally touching to each other at the point O_1 (as shown in the figure 6 above) hence, we have

$$\boldsymbol{Distance\ between\ the\ centres=C_1C_2=R_1+R_2}$$

But, we also have

$$\boldsymbol{C_1C_2 = PC_1 - PC_2} = \frac{R_1 \tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} - \frac{R_2 \tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}}$$

Now, equating both the above values of C_1C_2 as follows

$$R_1 + R_2 = \frac{R_1 \tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} - \frac{R_2 \tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} \quad \Rightarrow R_2\left(\frac{\tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} + 1\right) = R_1\left(\frac{\tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}} - 1\right)$$

$$\Rightarrow \quad R_2 = R_1\left(\frac{\tan\frac{\pi}{n} - \tan\frac{\alpha}{2}}{\tan\frac{\pi}{n} + \tan\frac{\alpha}{2}}\right) = R_1\left(\frac{\sin\frac{\pi}{n}\cos\frac{\alpha}{2} - \cos\frac{\pi}{n}\sin\frac{\alpha}{2}}{\sin\frac{\pi}{n}\cos\frac{\alpha}{2} + \cos\frac{\pi}{n}\sin\frac{\alpha}{2}}\right) = R_1\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)$$

$$\Rightarrow \quad \boldsymbol{R_2 = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)} \qquad (Since,\ \boldsymbol{R_1 = R = radius\ of\ the\ largest\ sphere})$$

Similarly, we can obtain radius R_3 of the sphere with centre C_3 externally touching the sphere with radius R_2 & centre C_2 as follows

$$R_3 = R_2\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right) = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right) = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)^2$$

$$\Rightarrow \quad \boldsymbol{R_3 = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)^2}$$

$$\Rightarrow \quad \boldsymbol{R_4 = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)^3}$$

$$\Rightarrow \quad \boldsymbol{R_5 = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)^4}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

$$\Rightarrow \quad \boldsymbol{R_{N-1} = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)^{N-2}}$$

$$\Rightarrow \quad \boldsymbol{R_N = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)^{N-1}}$$

Hence, the **radius ($\boldsymbol{R_N}$) of $\boldsymbol{N^{th}}$ snugly fitted (packed) sphere in the vertex of the right pyramid** is given as

$$\boldsymbol{R_N = R\left(\frac{\sin\left(\frac{\pi}{n} - \frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n} + \frac{\alpha}{2}\right)}\right)^{N-1}} \qquad \forall\ N \geq 1,\ \ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3$$

Where, R is the radius of the largest sphere inscribed in the right pyramid with regular n-gonal base & α is the angle between any two consecutive lateral edges out of total n no. of edges meeting at the vertex. It is equally applicable on regular bi-pyramids & all five platonic solids. **Note:** $N = 1$ is taken for largest inscribed sphere.

But we know that

$$\mathbf{0} < \left(\frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}\right) < \mathbf{1} \quad \forall\ n \geq 3\ \ \&\ \ \alpha < \frac{2\pi}{n}$$

The above generalised formula shows that the radii of the snugly fitted (packed) spheres decrease successively in a geometric progression (having a positive common ratio less than unity).

The total volume ($\left(V_{packed}\right)_N$) packed/occupied by all N no. of snugly fitted (packed) spheres in the right pyramid: There are total N no. of the spheres snugly fitted (packed) in the right pyramid, hence total volume of all the spheres with radii $R_1, R_2, R_3, \ldots\ldots\ldots. R_{N-1}, R_N$ is given as

$$\left(\boldsymbol{V}_{\boldsymbol{packed}}\right)_{\boldsymbol{N}} = \frac{\mathbf{4}\boldsymbol{\pi}}{\mathbf{3}}{\boldsymbol{R}_{\mathbf{1}}}^{\mathbf{3}} + \frac{\mathbf{4}\boldsymbol{\pi}}{\mathbf{3}}{\boldsymbol{R}_{\mathbf{2}}}^{\mathbf{3}} + \frac{\mathbf{4}\boldsymbol{\pi}}{\mathbf{3}}{\boldsymbol{R}_{\mathbf{3}}}^{\mathbf{3}} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + \frac{\mathbf{4}\boldsymbol{\pi}}{\mathbf{3}}{\boldsymbol{R}_{\boldsymbol{N}-\mathbf{1}}}^{\mathbf{3}} + \frac{\mathbf{4}\boldsymbol{\pi}}{\mathbf{3}}{\boldsymbol{R}_{\boldsymbol{N}}}^{\mathbf{3}}$$

$$we\ know, \quad \boldsymbol{R}_{\boldsymbol{N}} = \boldsymbol{R}\left(\frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}\right)^{\boldsymbol{N}-\mathbf{1}} = \boldsymbol{K}^{\boldsymbol{N}-\mathbf{1}}\boldsymbol{R} \quad Where,\ \boldsymbol{K} = \frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)} < \mathbf{1}$$

Now, setting the values of all the radii in term of R as follows

$$\left(V_{packed}\right)_N = \frac{4\pi}{3}R^3 + \frac{4\pi}{3}(KR)^3 + \frac{4\pi}{3}(K^2R)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + \frac{4\pi}{3}(K^{N-2}R)^3 + \frac{4\pi}{3}(K^{N-1}R)^3$$

$$= \frac{4\pi}{3}R^3\left(1 + K^3 + K^6 + K^9 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + K^{3(N-2)} + K^{3(N-1)}\right)$$

$$= \frac{4\pi}{3}R^3(sum\ of\ N\ terms\ of\ a\ geometric\ progression\ with\ a\ common\ ratio\ K^3)$$

$$= \frac{4\pi}{3}R^3\left(\frac{1(1-(K^3)^N)}{1-K^3}\right) = \frac{4\pi}{3}R^3\left(\frac{1-K^{3N}}{1-K^3}\right) \qquad (since,\ \ 0 < K < 1)$$

Hence, the **total volume packed/occupied all N no. of snugly fitted (packed) spheres in the right pyramid** is given as follows

$$\left(\boldsymbol{V}_{\boldsymbol{packed}}\right)_{\boldsymbol{N}} = \frac{\mathbf{4}\boldsymbol{\pi}}{\mathbf{3}}\boldsymbol{R}^{\mathbf{3}}\left(\frac{\mathbf{1}-\boldsymbol{K}^{\mathbf{3}\boldsymbol{N}}}{\mathbf{1}-\boldsymbol{K}^{\mathbf{3}}}\right)$$

$$Where\ \ \boldsymbol{packing\ constant},\ \boldsymbol{K} = \frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)} \qquad \left(\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3\ \ \Rightarrow\ \ \mathbf{0} < \boldsymbol{K} < \mathbf{1}\right)$$

The total volume ($\left(V_{packed}\right)_\infty$) packed/occupied by infinite no. of snugly fitted spheres in the right pyramid ($N \to \infty$): Taking the limit of the total volume packed/occupied by N no. of snugly fitted/packed spheres at $N \to \infty$ as follows

$$\left(V_{packed}\right)_\infty = \lim_{N\to\infty}\frac{4\pi}{3}R^3\left(\frac{1-K^{3N}}{1-K^3}\right) = \frac{4\pi}{3}R^3\lim_{N\to\infty}\left(\frac{1-K^{3N}}{1-K^3}\right) = \frac{4\pi}{3}R^3\left(\frac{1-0}{1-K^3}\right) \quad (since,\ \ 0 < K < 1\ )$$

$$\Rightarrow \left(\boldsymbol{V}_{\boldsymbol{packed}}\right)_{\infty} = \frac{\mathbf{4}\boldsymbol{\pi}\boldsymbol{R}^{\mathbf{3}}}{\mathbf{3}(\mathbf{1}-\boldsymbol{K}^{\mathbf{3}})}$$

The above volume $\left(\boldsymbol{V_{packed}}\right)_{\infty}$ is also called the **maximum packed volume.**

Packing ratio $\left(r_p\right)_N$ (i.e. the ratio of the total volume packed/occupied by all N no. of snugly fitted spheres to the volume of the right pyramid): The packing ratio ($\left(r_p\right)_N$) is given as follows

$$\left(r_p\right)_N=\frac{total\ volume\ occupied\ by\ all\ n\ no.of\ snugly\ fitted\ spheres}{volume\ of\ right\ pyramid}=\frac{\left(V_{packed}\right)_N}{V_{\Delta}}$$

$$=\frac{\frac{4\pi}{3}R^3\left(\frac{1-K^{3N}}{1-K^3}\right)}{\frac{1}{3}nR^3\left(\frac{\sin\frac{\pi}{n}sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{sin\frac{\alpha}{2}cos^2\frac{\pi}{n}\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}\right)}=4\left(\frac{\pi}{n}\right)\left(\frac{1-K^{3N}}{1-K^3}\right)\left(\frac{sin\frac{\alpha}{2}cos^2\frac{\pi}{n}sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\frac{\pi}{n}sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}\right)$$

$$\boldsymbol{\left(r_p\right)_N=4\left(\frac{\pi}{n}\right)\left(\frac{1-K^{3N}}{1-K^3}\right)\left(\frac{sin\frac{\alpha}{2}cos^2\frac{\pi}{n}sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\frac{\pi}{n}sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}\right)}$$

Packing ratio $\left(r_p\right)_{\infty}$ (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres to the volume of the right pyramid): The packing ratio ($\left(r_p\right)_{\infty}$) is given as follows

$$\left(r_p\right)_{\infty}=\frac{total\ volume\ occupied\ by\ infinite\ no.of\ snugly\ fitted\ spheres}{volume\ of\ right\ pyramid}=\frac{\left(V_{packed}\right)_{\infty}}{V_{\Delta}}$$

$$=\frac{\frac{4\pi R^3}{3(1-K^3)}}{\frac{1}{3}nR^3\left(\frac{\sin\frac{\pi}{n}sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}{sin\frac{\alpha}{2}cos^2\frac{\pi}{n}\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}\right)}=\frac{4\pi}{n(1-K^3)}\left(\frac{sin\frac{\alpha}{2}cos^2\frac{\pi}{n}sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\frac{\pi}{n}sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}\right)$$

$$\boldsymbol{\left(r_p\right)_{\infty}=\frac{4\pi}{n(1-K^3)}\left(\frac{sin\frac{\alpha}{2}cos^2\frac{\pi}{n}sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\frac{\pi}{n}sin^2\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}\right)}$$

$$Where\ \ \boldsymbol{packing\ constant},\ \boldsymbol{K}=\frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\right)}\qquad\left(\forall\ \alpha<\frac{2\pi}{n}\ \ \&\ \ n\geq 3\Rightarrow\ \ \boldsymbol{0<K<1}\right)$$

The above ratio $\left(\boldsymbol{r_p}\right)_{\infty}$ is also called the **maximum packing ratio.**

6. Packing of the spheres in the platonic solids: Let there be a platonic solid having $\boldsymbol{n_V}$ **no. of identical vertices,** $\boldsymbol{n}$ **no. of edges** meeting at each vertex such that angle between any two consecutive edges is $\boldsymbol{\alpha}$. If $\boldsymbol{R_i}$ is the **radius of the largest sphere inscribed in the platonic solid** having volume $\boldsymbol{V_s}$. In case of a platonic solid, N no. of spheres are snugly fitted (packed) in each of n_V no. of identical vertices excluding the largest inscribed sphere with radius R_i .Thus, the **total ($\boldsymbol{n_V N+1}$) no. of spheres are snugly fitted in a platonic solid**. All the **generalised formulae of right pyramid** are **slightly modified** for **platonic solids** simply by substituting $\boldsymbol{N=N+1}$ & $\boldsymbol{R=R_i}=radius\ of\ the\ \boldsymbol{largest\ inscribed\ sphere}$ in all the formula of sphere packing.

The radius (R_N) of the N^{th} sphere snugly fitted (packed) in one of n_V no. of identical vertex of a platonic solid (**excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one**) is given as follows

$$\boldsymbol{R_N = R_i\left(\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}\right)^N}$$

The total volume ($\left(V_{packed}\right)_N$) packed/occupied by all N no. of snugly fitted (packed) spheres in a platonic solid: Since there are N no. of the spheres snugly fitted (packed) in each of n_V no. of identical vertices of a platonic solid (**excluding the largest inscribed sphere**) hence total volume of all $(n_V N+1)$ spheres, out of which each N no. of snugly fitted sphere have radii $R_1, R_2, R_3, \dots\dots\dots. R_{N-1}, R_N$ & the largest inscribed sphere has radius R_i , is given as

$$\left(\boldsymbol{V_{packed}}\right)_{\boldsymbol{N}} = (\boldsymbol{no. of\ identical\ vertices}) \times (\boldsymbol{total\ volume\ of\ N\ no. of\ snugly\ fitted\ spheres}) \\ + (\boldsymbol{volume\ of\ the\ largest\ inscribed\ sphere})$$

$$\Rightarrow \left(\boldsymbol{V_{packed}}\right)_{\boldsymbol{N}} = \boldsymbol{n_V}\left(\frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_1}^3 + \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_2}^3 + \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_3}^3 + \dots\dots\dots\dots\dots\dots\dots.. + \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_{N-1}}^3 + \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_N}^3\right) + \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^3$$

$$we\ know, \quad \boldsymbol{R_N = R_i\left(\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}\right)^N = K^N R_i} \quad Where,\ \boldsymbol{K = \frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)} < 1}$$

Now, setting the values of all the radii in term of R_i we get

$$\left(V_{packed}\right)_N = n_V\left(\frac{4\pi}{3}(KR_i)^3 + \frac{4\pi}{3}(K^2R_i)^3 + \frac{4\pi}{3}(K^3R_i)^3 + \dots\dots\dots\dots\dots\dots\dots.. + \frac{4\pi}{3}(K^{N-1}R_i)^3 \\ + \frac{4\pi}{3}(K^NR_i)^3\right) + \frac{4\pi}{3}{R_i}^3$$

$$= \frac{4\pi}{3}n_V K^3{R_i}^3\left(1 + K^3 + K^6 + K^9 + \dots\dots\dots\dots\dots\dots\dots.. + K^{3(N-2)} + K^{3(N-1)}\right) + \frac{4\pi}{3}{R_i}^3$$

$$= \frac{4\pi}{3}n_V K^3{R_i}^3(sum\ of\ N\ terms\ of\ a\ geometric\ progression\ with\ a\ common\ ratio\ K^3) + \frac{4\pi}{3}{R_i}^3$$

$$= \frac{4\pi}{3}n_V K^3{R_i}^3\left(\frac{1(1-(K^3)^N)}{1-K^3}\right) + \frac{4\pi}{3}{R_i}^3 \qquad (since,\ \ 0 < K < 1)$$

$$= \frac{4\pi}{3}n_V K^3{R_i}^3\left(\frac{1-K^{3N}}{1-K^3}\right) + \frac{4\pi}{3}{R_i}^3 = \frac{4\pi}{3}{R_i}^3\left\{1 + n_V K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

Hence, the **total volume packed/occupied all $(\boldsymbol{n_V N+1})$ no. of snugly fitted (packed) spheres in a platonic solid** is given as

$$\left(\boldsymbol{V_{packed}}\right)_{\boldsymbol{N}} = \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^3\left\{\boldsymbol{1 + n_V K^3\left(\frac{1-K^{3N}}{1-K^3}\right)}\right\}$$

$$Where\ \boldsymbol{packing\ constant},\ \boldsymbol{K} = \frac{\boldsymbol{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}}{\boldsymbol{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}} \qquad \left(\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3\ \ \Rightarrow\ \ \boldsymbol{0 < K < 1}\right)$$

The total volume ($\left(V_{packed}\right)_\infty$) packed/occupied by infinite no. of snugly fitted spheres in all $\boldsymbol{n_V}$ no. of identical vertices of a platonic solid (including the largest inscribed sphere) $(\boldsymbol{N \to \infty})$**:** Taking the limit of the total volume packed/occupied by $(n_V N+1)$ no. of snugly fitted/packed spheres at $N \to \infty$ as follows

$$\left(V_{packed}\right)_{\infty}=\lim_{N\to\infty}\frac{4\pi}{3}{R_i}^3\left\{1+n_VK^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}=\frac{4\pi}{3}{R_i}^3\lim_{N\to\infty}\left\{1+n_VK^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$=\frac{4\pi}{3}{R_i}^3\left(1+n_VK^3\left(\frac{1-0}{1-K^3}\right)\right)=\frac{4\pi}{3}{R_i}^3\left(1+\frac{n_VK^3}{1-K^3}\right)\quad (since,\ \ 0<K<1\ )$$

$$\boldsymbol{\left(V_{packed}\right)_{\infty}=\frac{4\pi}{3}{R_i}^3\left(1+\frac{n_VK^3}{1-K^3}\right)}$$

Packing ratio $\left(r_p\right)_N$ (i.e. the ratio of the total volume packed/occupied by all (n_VN+1) no. of snugly fitted spheres to the volume of the platonic solid): The packing ratio ($\left(r_p\right)_N$) is given as follows

$$\left(r_p\right)_N=\frac{total\ volume\ occupied\ by\ all\ (n_VN+1)\ no.of\ snugly\ fitted\ spheres}{volume\ of\ platonic\ solid}=\frac{\left(V_{packed}\right)_N}{V_s}$$

$$=\frac{\frac{4\pi}{3}{R_i}^3\left\{1+n_VK^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}}{V_s}=\frac{4\pi}{3V_s}{R_i}^3\left\{1+n_VK^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$\boldsymbol{\left(r_p\right)_N=\frac{4\pi}{3V_s}{R_i}^3\left\{1+n_VK^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}}$$

Packing ratio $\left(r_p\right)_{\infty}$ (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all n_V no. of identical vertices of a platonic solid (**including the largest inscribed sphere**)**:** The packing ratio ($\left(r_p\right)_{\infty}$) is given as follows

$$\left(r_p\right)_{\infty}=\frac{total\ volume\ occupied\ by\ infinite\ no.of\ snugly\ fitted\ spheres}{volume\ of\ platonic\ solid}=\frac{\left(V_{packed}\right)_{\infty}}{V_s}$$

$$=\frac{\frac{4\pi}{3}{R_i}^3\left(1+\frac{n_VK^3}{1-K^3}\right)}{V_s}=\frac{4\pi}{3V_s}{R_i}^3\left(1+\frac{n_VK^3}{1-K^3}\right)$$

$$\boldsymbol{\left(r_p\right)_{\infty}=\frac{4\pi}{3V_s}{R_i}^3\left(1+\frac{n_VK^3}{1-K^3}\right)}$$

$$Where\ \boldsymbol{packing\ constant},\ \ \boldsymbol{K}=\frac{\boldsymbol{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}}{\boldsymbol{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}}\qquad\left(\forall\ \alpha<\frac{2\pi}{n}\ \ \&\ \ n\geq3\ \Rightarrow\ \ \boldsymbol{0<K<1}\right)$$

Packing of the spheres in all five platonic solids: In order to determine the **total packed volume** & **packing ratio (fraction**) of all the **platonic solids** we will directly take the data from the **'table of the important parameters of all five platonic solids'** prepared by the author **H.C. Rajpoot** such as **(inner) radius of the (largest) inscribed sphere** & the **volume of the corresponding platonic solid** in the order.

1. Regular tetrahedron: Let there be a regular tetrahedron with edge length a. Then in this case we have

$$\boldsymbol{R_i}=\ radius\ of\ the\ (largest)\ inscribed\ sphere\ in\ a\ regular\ tetrahedron=\boldsymbol{\frac{a}{2\sqrt{6}}}$$

$$\boldsymbol{V_s} = \ volume\ of\ a\ regular\ tetrahedron = \frac{\boldsymbol{a^3}}{\boldsymbol{6\sqrt{2}}}$$

$$\boldsymbol{n_V} = no.of\ identical\ vertices\ in\ a\ regular\ tetrahedron = \boldsymbol{4}$$

$$\boldsymbol{n} = no.of\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ tetrahedron = \boldsymbol{3}$$

$$\boldsymbol{\alpha} = angle\ between\ any\ two\ consecutive\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ tetrahedron$$
$$= \boldsymbol{60^o} = \frac{\boldsymbol{\pi}}{\boldsymbol{3}}$$

Hence, the **packing constant K of a regular tetrahedron** is calculated as follows

$$\boldsymbol{K} = \frac{\boldsymbol{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}}{\boldsymbol{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}} = \frac{\sin\left(\frac{\pi}{3}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)} = \frac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{1}{2} \quad \Rightarrow \quad \boxed{\boldsymbol{K} = \frac{\boldsymbol{1}}{\boldsymbol{2}}}$$

The **radius (R_N) of the N^{th} sphere** snugly fitted (packed) in each of $n_V = 4$ identical vertex of a regular tetrahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given as follows

$$\boldsymbol{R_N} = \boldsymbol{R_i}\left(\frac{\boldsymbol{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}}{\boldsymbol{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}}\right)^{\boldsymbol{N}} \quad \Rightarrow \quad \boldsymbol{R_N} = \frac{\boldsymbol{a}}{\boldsymbol{2\sqrt{6}}}\left(\frac{\boldsymbol{1}}{\boldsymbol{2}}\right)^{\boldsymbol{N}}$$

If there are N no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of $n_V = 4$ identical vertices of a regular tetrahedron then the total volume occupied by all $n_V N + 1 = 4N + 1$ snugly fitted spheres (including the largest inscribed sphere) is given as

$$\left(\boldsymbol{V_{packed}}\right)_{\boldsymbol{N}} = \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^{\boldsymbol{3}}\left\{\boldsymbol{1} + \boldsymbol{n_V K^3}\left(\frac{\boldsymbol{1-K^{3N}}}{\boldsymbol{1-K^3}}\right)\right\} = \frac{4\pi}{3}\left(\frac{a}{2\sqrt{6}}\right)^3\left\{1 + (4)\left(\frac{1}{2}\right)^3\left(\frac{1-\left(\frac{1}{2}\right)^{3N}}{1-\left(\frac{1}{2}\right)^3}\right)\right\}$$

$$\therefore \ \left(\boldsymbol{V_{packed}}\right)_{\boldsymbol{N}} = \frac{\boldsymbol{\pi a^3}}{\boldsymbol{36\sqrt{6}}}\left\{\boldsymbol{1} + \frac{\boldsymbol{4}}{\boldsymbol{7}}\left(\frac{\boldsymbol{2^{3N}-1}}{\boldsymbol{2^{3N}}}\right)\right\}$$

The **total volume ($\left(V_{packed}\right)_{\infty}$)** packed/occupied by infinite no. of snugly fitted spheres in all $n_V = 4$ identical vertices of a regular tetrahedron (including the largest inscribed sphere) $(N \to \infty)$ is given as

$$\left(\boldsymbol{V_{packed}}\right)_{\infty} = \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^{\boldsymbol{3}}\left(\boldsymbol{1} + \frac{\boldsymbol{n_V K^3}}{\boldsymbol{1-K^3}}\right) = \frac{4\pi}{3}\left(\frac{a}{2\sqrt{6}}\right)^3\left(1 + \frac{4\left(\frac{1}{2}\right)^3}{1-\left(\frac{1}{2}\right)^3}\right) = \frac{\pi a^3}{36\sqrt{6}}\left(1+\frac{4}{7}\right) = \frac{11\pi a^3}{252\sqrt{6}}$$

$$\therefore \ \left(\boldsymbol{V_{packed}}\right)_{\infty} = \frac{\boldsymbol{11\pi a^3}}{\boldsymbol{252\sqrt{6}}}$$

Packing ratio $\left(\boldsymbol{r_p}\right)_{\boldsymbol{N}}$ (i.e. the ratio of the total volume packed/occupied by all $(4N + 1)$ no. of snugly fitted spheres to the volume of the regular tetrahedron): The packing ratio ($\left(r_p\right)_N$) is given as follows

$$\left(r_p\right)_N = \frac{total\ volume\ occupied\ by\ all\ (4N+1)\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ tetrahedron} = \frac{\left(V_{packed}\right)_N}{V_s}$$

$$\Rightarrow \left(r_p\right)_N=\frac{\frac{\pi a^3}{36\sqrt{6}}\left\{1+\frac{4}{7}\left(\frac{2^{3N}-1}{2^{3N}}\right)\right\}}{\frac{a^3}{6\sqrt{2}}}=\frac{\pi}{6\sqrt{3}}\left\{1+\frac{4}{7}\left(\frac{2^{3N}-1}{2^{3N}}\right)\right\}$$

$$\therefore\ \boldsymbol{\left(r_p\right)_N=\frac{\pi}{6\sqrt{3}}\left\{1+\frac{4}{7}\left(\frac{2^{3N}-1}{2^{3N}}\right)\right\}}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_N$ depends only on N no. of spheres snugly fitted/packed in each of $n_V=4$ identical vertices of a regular tetrahedron.

Packing ratio $\left(r_p\right)_\infty$ (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all $n_V=4$ identical vertices of a regular tetrahedron (including the largest inscribed sphere): The packing ratio ($\left(r_p\right)_\infty$) is given as follows

$$\left(r_p\right)_\infty=\frac{total\ volume\ occupied\ by\ infinite\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ tetrahedron}=\frac{\left(V_{packed}\right)_\infty}{V_s}$$

$$\Rightarrow \left(r_p\right)_\infty=\frac{\frac{11\pi a^3}{252\sqrt{6}}}{\frac{a^3}{6\sqrt{2}}}=\frac{11\pi}{42\sqrt{3}}$$

$$\therefore\ \boldsymbol{\left(r_p\right)_\infty=\frac{11\pi}{42\sqrt{3}}\approx 0.47504269}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_\infty$ is the maximum possible value which shows that approximate 47.5 % of the volume of any regular tetrahedron can be packed by snugly fitting infinite no. of the spheres in each of its four identical vertices including the (volume of) largest inscribed sphere touching all four triangular faces.

2. Regular hexahedron (cube): Let there be a regular hexahedron (cube) with edge length a. In this case we have

$$\boldsymbol{R_i}=\ radius\ of\ the\ (largest)\ inscribed\ sphere\ in\ a\ regular\ hexahedron=\boldsymbol{\frac{a}{2}}$$

$$\boldsymbol{V_s}=\ volume\ of\ a\ regular\ hexahedron=\boldsymbol{a^3}$$

$$\boldsymbol{n_V}=no.of\ identical\ vertices\ in\ a\ regular\ hexahedron=\boldsymbol{8}$$

$$\boldsymbol{n}=no.of\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ hexahedron=\boldsymbol{3}$$

$$\boldsymbol{\alpha}=angle\ between\ any\ two\ consecutive\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ hexahedron=\boldsymbol{90^o}=\boldsymbol{\frac{\pi}{2}}$$

Hence, the **packing constant K of a regular hexahedron (cube)** is calculated as follows

$$\boldsymbol{K}=\frac{\boldsymbol{sin}\left(\boldsymbol{\frac{\pi}{n}-\frac{\alpha}{2}}\right)}{\boldsymbol{sin}\left(\boldsymbol{\frac{\pi}{n}+\frac{\alpha}{2}}\right)}=\frac{sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)}{sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)}=\frac{sin\left(\frac{\pi}{12}\right)}{cos\left(\frac{\pi}{12}\right)}=tan\left(\frac{\pi}{12}\right)=2-\sqrt{3}\quad\Rightarrow\quad\boxed{\boldsymbol{K=2-\sqrt{3}}}$$

The **radius (R_N) of the N^{th} sphere** snugly fitted (packed) in each of $n_V=8$ identical vertex of a regular hexahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given as follows

$$\boldsymbol{R_N = R_i\left(\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}\right)^N} \quad\Rightarrow\quad \boldsymbol{R_N = \frac{a}{2}\left(2-\sqrt{3}\right)^N}$$

If there are N no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of $n_V = 8$ identical vertices of a regular hexahedron then the **total volume occupied** by all $n_V N + 1 = 8N + 1$ snugly fitted spheres (including the largest inscribed sphere) is given as

$$\left(\boldsymbol{V_{packed}}\right)_N = \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^3\left\{\boldsymbol{1 + n_V K^3\left(\frac{1-K^{3N}}{1-K^3}\right)}\right\} = \frac{4\pi}{3}\left(\frac{a}{2}\right)^3\left\{1 + (8)\left(2-\sqrt{3}\right)^3\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{1-\left(2-\sqrt{3}\right)^3}\right)\right\}$$

$$= \frac{\pi a^3}{6}\left\{1 + 8\left(26-15\sqrt{3}\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{15\sqrt{3}-25}\right)\right\} = \frac{\pi a^3}{6}\left\{1 + \frac{8}{5}\left(26-15\sqrt{3}\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{3\sqrt{3}-5}\right)\right\}$$

$$= \frac{\pi a^3}{6}\left\{1 + \frac{8}{5}\left(26-15\sqrt{3}\right)\left(3\sqrt{3}+5\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{\left(3\sqrt{3}-5\right)\left(3\sqrt{3}+5\right)}\right)\right\}$$

$$= \frac{\pi a^3}{6}\left\{1 + \frac{8}{5}\left(3\sqrt{3}-5\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{2}\right)\right\} = \frac{\pi a^3}{6}\left\{1 + \frac{4}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$$

$$\therefore\ \left(\boldsymbol{V_{packed}}\right)_N = \boldsymbol{\frac{\pi a^3}{6}\left\{1 + \frac{4}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}}$$

The total volume ($\left(V_{packed}\right)_\infty$) packed/occupied by infinite no. of snugly fitted spheres in all $n_V = 8$ identical vertices of a regular hexahedron (including the largest inscribed sphere) $(N \to \infty)$ is given as

$$\left(\boldsymbol{V_{packed}}\right)_\infty = \frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^3\left(\boldsymbol{1 + \frac{n_V K^3}{1-K^3}}\right) = \frac{4\pi}{3}\left(\frac{a}{2}\right)^3\left(1 + \frac{8\left(2-\sqrt{3}\right)^3}{1-\left(2-\sqrt{3}\right)^3}\right) = \frac{\pi a^3}{6}\left(1 + \frac{8\left(26-15\sqrt{3}\right)}{15\sqrt{3}-25}\right)$$

$$= \frac{\pi a^3}{6}\left(1 + \frac{4\left(3\sqrt{3}-5\right)}{5}\right) = \frac{\pi a^3}{6}\left(\frac{12\sqrt{3}-15}{5}\right) = \pi a^3\left(\frac{4\sqrt{3}-5}{10}\right)$$

$$\therefore\ \left(\boldsymbol{V_{packed}}\right)_\infty = \boldsymbol{\pi a^3\left(\frac{4\sqrt{3}-5}{10}\right)}$$

Packing ratio $\left(\boldsymbol{r_p}\right)_N$ (i.e. the ratio of the total volume packed/occupied by all $(8N+1)$ no. of snugly fitted spheres to the volume of the regular hexahedron/cube): The packing ratio ($\left(r_p\right)_N$) is given as follows

$$\left(r_p\right)_N = \frac{total\ volume\ occupied\ by\ all\ (8N+1)\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ hexahedron} = \frac{\left(V_{packed}\right)_N}{V_s}$$

$$\Rightarrow \left(r_p\right)_N = \frac{\frac{\pi a^3}{6}\left\{1 + \frac{4}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}}{a^3} = \frac{\pi}{6}\left\{1 + \frac{4}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$$

$$\therefore\ \left(\boldsymbol{r_p}\right)_N = \boldsymbol{\frac{\pi}{6}\left\{1 + \frac{4}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_N$ depends only on N no. of spheres snugly fitted/packed in each of $n_V = 8$ identical vertices of a regular hexahedron (cube).

Packing ratio $\left(\boldsymbol{r_p}\right)_\infty$ (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all $n_V = 8$ identical vertices of a regular hexahedron (including the largest inscribed sphere): The packing ratio ($\left(r_p\right)_\infty$) is given as follows

$$\left(r_p\right)_\infty = \frac{total\ volume\ occupied\ by\ infinite\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ hexahedron} = \frac{\left(V_{packed}\right)_\infty}{V_s}$$

$$\Rightarrow \left(r_p\right)_\infty = \frac{\pi a^3\left(\frac{4\sqrt{3}-5}{10}\right)}{a^3} = \pi\left(\frac{4\sqrt{3}-5}{10}\right)$$

$$\therefore\ \left(\boldsymbol{r_p}\right)_\infty = \boldsymbol{\pi}\left(\frac{\mathbf{4\sqrt{3}-5}}{\mathbf{10}}\right) \approx \mathbf{0.60576291}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_\infty$ is the maximum possible value which shows that approximate 60.58 % of the volume of any regular hexahedron (cube) can be packed by snugly fitting infinite no. of the spheres in each of its eight identical vertices including the (volume of) largest inscribed sphere touching all six square faces.

3. Regular octahedron: Let there be a regular octahedron with edge length a. In this case we have

$$\boldsymbol{R_i} = \ radius\ of\ the\ (largest)\ inscribed\ sphere\ in\ a\ regular\ octahedron = \frac{\boldsymbol{a}}{\sqrt{\mathbf{6}}}$$

$$\boldsymbol{V_s} = \ volume\ of\ a\ regular\ octahedron = \frac{\boldsymbol{a^3\sqrt{2}}}{\mathbf{3}}$$

$$\boldsymbol{n_V} = no.of\ identical\ vertices\ in\ a\ regular\ octahedron = \mathbf{6}$$

$$\boldsymbol{n} = no.of\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ octahedron = \mathbf{4}$$

$$\boldsymbol{\alpha} = angle\ between\ any\ two\ consecutive\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ octahedron$$
$$= \mathbf{60^o} = \frac{\boldsymbol{\pi}}{\mathbf{3}}$$

Hence, the **packing constant $\boldsymbol{K}$ of a regular octahedron** is calculated as follows

$$\boldsymbol{K} = \frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)} = \frac{\sin\left(\frac{\pi}{4}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)} = \frac{\sin\left(\frac{\pi}{12}\right)}{\cos\left(\frac{\pi}{12}\right)} = \tan\left(\frac{\pi}{12}\right) = 2-\sqrt{3} \qquad \Rightarrow \boxed{\boldsymbol{K} = \mathbf{2}-\sqrt{\mathbf{3}}}$$

The **radius** $(\boldsymbol{R_N})$ **of the** $\boldsymbol{N^{th}}$ **sphere** snugly fitted (packed) in each of $n_V = 6$ identical vertex of a regular octahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given as follows

$$\boldsymbol{R_N} = \boldsymbol{R_i}\left(\frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}\right)^{\boldsymbol{N}} \quad \Rightarrow \quad \boldsymbol{R_N} = \frac{\boldsymbol{a}}{\sqrt{\mathbf{6}}}\left(\mathbf{2}-\sqrt{\mathbf{3}}\right)^{\boldsymbol{N}}$$

If there are N no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of $n_V = 6$ identical vertices of a regular octahedron then the **total volume occupied** by all $n_V N + 1 = 6N + 1$ snugly fitted spheres (including the largest inscribed sphere) is given as

$$\left(V_{packed}\right)_N=\frac{4\pi}{3}{R_i}^3\left\{1+n_VK^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}=\frac{4\pi}{3}\left(\frac{a}{\sqrt{6}}\right)^3\left\{1+(6)\left(2-\sqrt{3}\right)^3\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{1-\left(2-\sqrt{3}\right)^3}\right)\right\}$$

$$=\frac{2\pi a^3}{9\sqrt{6}}\left\{1+6\left(26-15\sqrt{3}\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{15\sqrt{3}-25}\right)\right\}=\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left\{1+\frac{6}{5}\left(26-15\sqrt{3}\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{3\sqrt{3}-5}\right)\right\}$$

$$=\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left\{1+\frac{6}{5}\left(26-15\sqrt{3}\right)\left(3\sqrt{3}+5\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{\left(3\sqrt{3}-5\right)\left(3\sqrt{3}+5\right)}\right)\right\}$$

$$=\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left\{1+\frac{6}{5}\left(3\sqrt{3}-5\right)\left(\frac{1-\left(2-\sqrt{3}\right)^{3N}}{2}\right)\right\}=\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left\{1+\frac{3}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$$

$$\therefore\ \left(V_{packed}\right)_N=\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left\{1+\frac{3}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$$

The total volume ($\left(V_{packed}\right)_\infty$) packed/occupied by infinite no. of snugly fitted spheres in all $n_V=6$ identical vertices of a regular octahedron (including the largest inscribed sphere) $(N\to\infty)$ is given as

$$\left(V_{packed}\right)_\infty=\frac{4\pi}{3}{R_i}^3\left(1+\frac{n_VK^3}{1-K^3}\right)=\frac{4\pi}{3}\left(\frac{a}{\sqrt{6}}\right)^3\left(1+\frac{6\left(2-\sqrt{3}\right)^3}{1-\left(2-\sqrt{3}\right)^3}\right)=\frac{2\pi a^3}{9\sqrt{6}}\left(1+\frac{6\left(26-15\sqrt{3}\right)}{15\sqrt{3}-25}\right)$$

$$=\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left(1+\frac{3\left(3\sqrt{3}-5\right)}{5}\right)=\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left(\frac{9\sqrt{3}-10}{5}\right)=\pi a^3\sqrt{\frac{2}{3}}\left(\frac{9\sqrt{3}-10}{45}\right)$$

$$\therefore\ \left(V_{packed}\right)_\infty=\pi a^3\sqrt{\frac{2}{3}}\left(\frac{9\sqrt{3}-10}{45}\right)$$

Packing ratio $\left(r_p\right)_N$ (i.e. the ratio of the total volume packed/occupied by all $(6N+1)$ no. of snugly fitted spheres to the volume of the regular octahedron): The packing ratio ($\left(r_p\right)_N$) is given as follows

$$\left(r_p\right)_N=\frac{total\ volume\ occupied\ by\ all\ (6N+1)\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ octahedron}=\frac{\left(V_{packed}\right)_N}{V_s}$$

$$\Rightarrow\ \left(r_p\right)_N=\frac{\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left\{1+\frac{3}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}}{\frac{a^3\sqrt{2}}{3}}=\frac{\pi}{3\sqrt{3}}\left\{1+\frac{3}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$$

$$\therefore\ \left(r_p\right)_N=\frac{\pi}{3\sqrt{3}}\left\{1+\frac{3}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_N$ depends only on N no. of spheres snugly fitted/packed in each of $n_V=6$ identical vertices of a regular octahedron.

Packing ratio $\left(r_p\right)_\infty$ (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all $n_V=6$ identical vertices of a regular octahedron (including the largest inscribed sphere): The packing ratio ($\left(r_p\right)_\infty$) is given as follows

$$\left(r_p\right)_\infty = \frac{total\ volume\ occupied\ by\ infinite\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ octahedron} = \frac{\left(V_{packed}\right)_\infty}{V_s}$$

$$\Rightarrow \left(r_p\right)_\infty = \frac{\pi a^3\sqrt{\frac{2}{3}}\left(\frac{9\sqrt{3}-10}{45}\right)}{\frac{a^3\sqrt{2}}{3}} = \pi\left(\frac{9\sqrt{3}-10}{15\sqrt{3}}\right)$$

$$\therefore\ \left(\boldsymbol{r_p}\right)_\infty = \boldsymbol{\pi}\left(\frac{\mathbf{9\sqrt{3}-10}}{\mathbf{15\sqrt{3}}}\right) \approx \mathbf{0.675756016}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_\infty$ is the maximum possible value which shows that approximate 67.57 % of the volume of any regular octahedron can be packed by snugly fitting infinite no. of the spheres in each of its six identical vertices including the (volume of) largest inscribed sphere touching all eight triangular faces.

4. Regular dodecahedron: Let there be a regular dodecahedron with edge length a. In this case we have

$$\boldsymbol{R_i} = radius\ of\ the\ (largest) inscribed\ sphere\ in\ a\ regular\ dodecahedron = \frac{\boldsymbol{a}(\mathbf{3}+\sqrt{\mathbf{5}})}{\mathbf{2}\sqrt{\mathbf{10}-\mathbf{2}\sqrt{\mathbf{5}}}}$$

$$= \frac{\boldsymbol{a}}{\mathbf{2}}\sqrt{\frac{\mathbf{25}+\mathbf{11}\sqrt{\mathbf{5}}}{\mathbf{10}}}$$

$$\boldsymbol{V_s} = volume\ of\ a\ regular\ dodecahedron = \frac{\boldsymbol{a}^{\mathbf{3}}(\mathbf{15}+\mathbf{7}\sqrt{\mathbf{5}})}{\mathbf{4}}$$

$$\boldsymbol{n_V} = no.of\ identical\ vertices\ in\ a\ regular\ dodecahedron = \mathbf{20}$$

$$\boldsymbol{n} = no.of\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ dodecahedron = \mathbf{3}$$

$$\boldsymbol{\alpha} = angle\ between\ any\ two\ consecutive\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ dodecahedron$$

$$= \mathbf{108^o} = \frac{\mathbf{3\pi}}{\mathbf{5}}$$

Hence, the **packing constant K of a regular dodecahedron** is calculated as follows

$$\boldsymbol{K} = \frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)} = \frac{sin\left(\frac{\pi}{3}-\frac{3\pi}{10}\right)}{sin\left(\frac{\pi}{3}+\frac{3\pi}{10}\right)} = \frac{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}{4} \quad \Rightarrow \quad \boxed{\boldsymbol{K} = \frac{\mathbf{11}+\mathbf{3}\sqrt{\mathbf{5}}-\sqrt{\mathbf{150}+\mathbf{66}\sqrt{\mathbf{5}}}}{\mathbf{4}}}$$

The **radius** $(\boldsymbol{R_N})$ **of the** $\boldsymbol{N^{th}}$ **sphere** snugly fitted (packed) in each of $n_V = 20$ identical vertex of a regular dodecahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given as follows

$$\boldsymbol{R_N} = \boldsymbol{R_i}\left(\frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}\right)^{\boldsymbol{N}} \quad \Rightarrow \quad \boldsymbol{R_N} = \frac{\boldsymbol{a}}{\mathbf{2}}\sqrt{\frac{\mathbf{25}+\mathbf{11}\sqrt{\mathbf{5}}}{\mathbf{10}}}\left(\frac{\mathbf{11}+\mathbf{3}\sqrt{\mathbf{5}}-\sqrt{\mathbf{150}+\mathbf{66}\sqrt{\mathbf{5}}}}{\mathbf{4}}\right)^{\boldsymbol{N}}$$

If there are N no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of $n_V = 20$ identical vertices of a regular dodecahedron then the **total volume occupied** by all $n_V N + 1 = 20N + 1$ snugly fitted spheres (including the largest inscribed sphere) is given as

$$\left(\boldsymbol{V_{packed}}\right)_N=\frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^3\left\{\boldsymbol{1+n_V K^3}\left(\frac{\boldsymbol{1-K^{3N}}}{\boldsymbol{1-K^3}}\right)\right\}=\frac{4\pi}{3}\left(\frac{a}{2}\sqrt{\frac{25+11\sqrt{5}}{10}}\right)^3\left\{1+(20)K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$=\frac{\pi a^3}{60}\sqrt{\frac{61000+27280\sqrt{5}}{10}}\left\{1+20K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}=\frac{\pi a^3\sqrt{6100+2728\sqrt{5}}}{60}\left\{1+20K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$\therefore\ \left(\boldsymbol{V_{packed}}\right)_N=\frac{\boldsymbol{\pi a^3\sqrt{6100+2728\sqrt{5}}}}{\boldsymbol{60}}\left\{\boldsymbol{1+20K^3}\left(\frac{\boldsymbol{1-K^{3N}}}{\boldsymbol{1-K^3}}\right)\right\}$$

$$Where,\qquad \boldsymbol{packing\ constant,\ K}=\frac{\boldsymbol{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}}{\boldsymbol{4}}\approx\boldsymbol{0.114420648}$$

The total volume $\left(\left(\boldsymbol{V_{packed}}\right)_\infty\right)$ packed/occupied by infinite no. of snugly fitted spheres in all $n_V=20$ identical vertices of a regular dodecahedron (including the largest inscribed sphere) $(N\to\infty)$ is given as

$$\left(\boldsymbol{V_{packed}}\right)_\infty=\frac{\boldsymbol{4\pi}}{\boldsymbol{3}}\boldsymbol{R_i}^3\left(\boldsymbol{1+\frac{n_V K^3}{1-K^3}}\right)=\frac{4\pi}{3}\left(\frac{a}{2}\sqrt{\frac{25+11\sqrt{5}}{10}}\right)^3\left(1+\frac{20K^3}{1-K^3}\right)$$

$$=\frac{\pi a^3\sqrt{6100+2728\sqrt{5}}}{60}\left(\frac{1-K^3+20K^3}{1-K^3}\right)=\frac{\pi a^3\sqrt{6100+2728\sqrt{5}}}{60}\left(\frac{1+19K^3}{1-K^3}\right)$$

$$\therefore\ \left(\boldsymbol{V_{packed}}\right)_\infty=\frac{\boldsymbol{\pi a^3\sqrt{6100+2728\sqrt{5}}}}{\boldsymbol{60}}\left(\frac{\boldsymbol{1+19K^3}}{\boldsymbol{1-K^3}}\right)$$

Packing ratio $\left(\boldsymbol{r_p}\right)_N$ (i.e. the ratio of the total volume packed/occupied by all $(20N+1)$ no. of snugly fitted spheres to the volume of the regular dodecahedron): The packing ratio $\left((r_p)_N\right)$ is given as follows

$$(r_p)_N=\frac{total\ volume\ occupied\ by\ all\ (20N+1)\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ dodecahedron}=\frac{\left(V_{packed}\right)_N}{V_s}$$

$$\Rightarrow\ (r_p)_N=\frac{\frac{\pi a^3\sqrt{6100+2728\sqrt{5}}}{60}\left\{1+20K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}}{\frac{a^3(15+7\sqrt{5})}{4}}=\frac{\pi\sqrt{6100+2728\sqrt{5}}}{15(15+7\sqrt{5})}\left\{1+20K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$=\frac{\pi}{15}\sqrt{\frac{65+29\sqrt{5}}{10}}\left\{1+20K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$\therefore\ \left(\boldsymbol{r_p}\right)_N=\frac{\boldsymbol{\pi}}{\boldsymbol{15}}\sqrt{\frac{\boldsymbol{65+29\sqrt{5}}}{\boldsymbol{10}}}\left\{\boldsymbol{1+20K^3}\left(\frac{\boldsymbol{1-K^{3N}}}{\boldsymbol{1-K^3}}\right)\right\}$$

It is to be noted that the value of the packing ratio $(r_p)_N$ depends only on N no. of spheres snugly fitted/packed in each of $n_V=20$ identical vertices of a regular dodecahedron.

Packing ratio $\left(\boldsymbol{r_p}\right)_\infty$ (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all $n_V=20$ identical vertices of a regular dodecahedron (including the largest inscribed sphere): The packing ratio $\left((r_p)_\infty\right)$ is given as follows

$$\left(r_p\right)_{\infty}=\frac{total\ volume\ occupied\ by\ infinite\ no.of\ snugly\ fitted\ spheres}{volume\ of\ regular\ dodecahedron}=\frac{\left(V_{packed}\right)_{\infty}}{V_s}$$

$$\Rightarrow\ \left(r_p\right)_{\infty}=\frac{\frac{\pi a^3\sqrt{6100+2728\sqrt{5}}}{60}\left(\frac{1+19K^3}{1-K^3}\right)}{\frac{a^3\left(15+7\sqrt{5}\right)}{4}}=\frac{\pi\sqrt{6100+2728\sqrt{5}}}{15\left(15+7\sqrt{5}\right)}\left(\frac{1+19K^3}{1-K^3}\right)$$

$$\therefore\ \left(\boldsymbol{r_p}\right)_{\infty}=\frac{\boldsymbol{\pi}}{\mathbf{15}}\sqrt{\frac{\mathbf{65+29\sqrt{5}}}{\mathbf{10}}}\left(\frac{\mathbf{1+19}\boldsymbol{K}^3}{\mathbf{1-}\boldsymbol{K}^3}\right)\approx\mathbf{0.777342128}$$

$$Where,\qquad \boldsymbol{packing\ constant,}\ \ \boldsymbol{K}=\frac{\mathbf{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}}{\mathbf{4}}\approx\mathbf{0.114420648}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_{\infty}$ is the maximum possible value which shows that approximate 77.73 % of the volume of any regular dodecahedron can be packed by snugly fitting infinite no. of the spheres in each of its 20 identical vertices including the (volume of) largest inscribed sphere touching all 12 pentagonal faces.

5. Regular icosahedron: Let there be a regular icosahedron with edge length a. In this case we have

$$\boldsymbol{R_i}=\ radius\ of\ the\ (largest)inscribed\ sphere\ in\ a\ regular\ icosahedron=\frac{\boldsymbol{a}\left(\mathbf{3+\sqrt{5}}\right)}{\mathbf{4\sqrt{3}}}$$

$$\boldsymbol{V_s}=\ volume\ of\ a\ regular\ icosahedron=\frac{\mathbf{5}\boldsymbol{a}^3\left(\mathbf{3+\sqrt{5}}\right)}{\mathbf{12}}$$

$$\boldsymbol{n_V}=no.of\ identical\ vertices\ in\ a\ regular\ icosahedron=\mathbf{12}$$

$$\boldsymbol{n}=no.of\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ icosahedron=\mathbf{5}$$

$$\boldsymbol{\alpha}=angle\ between\ any\ two\ consecutive\ edges\ meeting\ at\ each\ vertex\ in\ a\ regular\ icosahedron$$
$$=\mathbf{60^o}=\frac{\boldsymbol{\pi}}{\mathbf{3}}$$

Hence, the **packing constant K of a regular icosahedron** is calculated as follows

$$\boldsymbol{K}=\frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}=\frac{\sin\left(\frac{\pi}{5}-\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{5}+\frac{\pi}{6}\right)}=\frac{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}{4}\quad\Rightarrow\quad\boxed{\boldsymbol{K}=\frac{\mathbf{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}}{\mathbf{4}}}$$

The **radius** ($\boldsymbol{R_N}$) **of the $\boldsymbol{N^{th}}$ sphere** snugly fitted (packed) in each of $n_V=12$ identical vertex of a regular icosahedron (excluding the largest inscribed sphere i.e. counting/sequence starts from the sphere just next to the largest one) is given as follows

$$\boldsymbol{R_N}=\boldsymbol{R_i}\left(\frac{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}-\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}{\boldsymbol{sin}\left(\frac{\boldsymbol{\pi}}{\boldsymbol{n}}+\frac{\boldsymbol{\alpha}}{\mathbf{2}}\right)}\right)^{\boldsymbol{N}}\quad\Rightarrow\quad\boldsymbol{R_N}=\frac{\boldsymbol{a}\left(\mathbf{3+\sqrt{5}}\right)}{\mathbf{4\sqrt{3}}}\left(\frac{\mathbf{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}}{\mathbf{4}}\right)^{\boldsymbol{N}}$$

If there are N no. of spheres (excluding the largest inscribed sphere) snugly fitted/packed in each of $n_V=12$ identical vertices of a regular icosahedron then the **total volume occupied** by all $n_VN+1=12N+1$ snugly fitted spheres (including the largest inscribed sphere) is given as

$$\boldsymbol{\left(V_{packed}\right)_N = \frac{4\pi}{3}R_i^{\,3}\left\{1+n_VK^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}} = \frac{4\pi}{3}\left(\frac{a\left(3+\sqrt{5}\right)}{4\sqrt{3}}\right)^3\left\{1+(12)K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$= \frac{\pi a^3\left(72+32\sqrt{5}\right)}{144\sqrt{3}}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\} = \frac{\pi a^3\left(9+4\sqrt{5}\right)}{18\sqrt{3}}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$\therefore\ \boldsymbol{\left(V_{packed}\right)_N = \frac{\pi a^3\left(9+4\sqrt{5}\right)}{18\sqrt{3}}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}}$$

$$Where, \qquad \boldsymbol{packing\ constant,\ K = \frac{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}{4} \approx 0.114420648}$$

The total volume ($\left(V_{packed}\right)_\infty$) packed/occupied by infinite no. of snugly fitted spheres in all $n_V = 12$ identical vertices of a regular icosahedron (including the largest inscribed sphere) $(N \to \infty)$ is given as

$$\boldsymbol{\left(V_{packed}\right)_\infty = \frac{4\pi}{3}R_i^{\,3}\left(1+\frac{n_VK^3}{1-K^3}\right)} = \frac{4\pi}{3}\left(\frac{a\left(3+\sqrt{5}\right)}{4\sqrt{3}}\right)^3\left(1+\frac{12K^3}{1-K^3}\right)$$

$$= \frac{\pi a^3\left(72+32\sqrt{5}\right)}{144\sqrt{3}}\left(\frac{1-K^3+12K^3}{1-K^3}\right) = \frac{\pi a^3\left(9+4\sqrt{5}\right)}{18\sqrt{3}}\left(\frac{1+11K^3}{1-K^3}\right)$$

$$\therefore\ \boldsymbol{\left(V_{packed}\right)_\infty = \frac{\pi a^3\left(9+4\sqrt{5}\right)}{18\sqrt{3}}\left(\frac{1+11K^3}{1-K^3}\right)}$$

Packing ratio $\boldsymbol{\left(r_p\right)_N}$ (i.e. the ratio of the total volume packed/occupied by all $(12N+1)$ no. of snugly fitted spheres to the volume of the regular icosahedron): The packing ratio ($\left(r_p\right)_N$) is given as follows

$$\left(r_p\right)_N = \frac{total\ volume\ occupied\ by\ all\ (12N+1)\ no. of\ snugly\ fitted\ spheres}{volume\ of\ regular\ icosahedron} = \frac{\left(V_{packed}\right)_N}{V_s}$$

$$\Rightarrow\ \left(r_p\right)_N = \frac{\dfrac{\pi a^3\left(9+4\sqrt{5}\right)}{18\sqrt{3}}\left\{1+12K^3\left(\dfrac{1-K^{3N}}{1-K^3}\right)\right\}}{\dfrac{5a^3\left(3+\sqrt{5}\right)}{12}} = \frac{2\pi\left(9+4\sqrt{5}\right)}{15\sqrt{3}\left(3+\sqrt{5}\right)}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$= \frac{\pi\left(7+3\sqrt{5}\right)}{30\sqrt{3}}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$$

$$\therefore\ \boldsymbol{\left(r_p\right)_N = \frac{\pi\left(7+3\sqrt{5}\right)}{30\sqrt{3}}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}}$$

It is to be noted that the value of the packing ratio $\left(r_p\right)_N$ depends only on N no. of spheres snugly fitted/packed in each of $n_V = 12$ identical vertices of a regular icosahedron.

Packing ratio $\boldsymbol{\left(r_p\right)_\infty}$ (i.e. the ratio of the total volume packed/occupied by infinite no. of snugly fitted spheres in all $n_V = 12$ identical vertices of a regular icosahedron (including the largest inscribed sphere): The packing ratio ($\left(r_p\right)_\infty$) is given as follows

$$\left(r_p\right)_\infty = \frac{total\ volume\ occupied\ by\ infinite\ no. of\ snugly\ fitted\ spheres}{volume\ of\ regular\ icosahedron} = \frac{\left(V_{packed}\right)_\infty}{V_s}$$

$$\Rightarrow (r_p)_{\infty} = \frac{\frac{\pi a^3(9+4\sqrt{5})}{18\sqrt{3}}\left(\frac{1+11K^3}{1-K^3}\right)}{\frac{5a^3(3+\sqrt{5})}{12}} = \frac{2\pi(9+4\sqrt{5})}{15\sqrt{3}(3+\sqrt{5})}\left(\frac{1+11K^3}{1-K^3}\right) = \frac{\pi(7+3\sqrt{5})}{30\sqrt{3}}\left(\frac{1+11K^3}{1-K^3}\right)$$

$$\therefore\ (r_p)_{\infty} = \frac{\pi(7+3\sqrt{5})}{30\sqrt{3}}\left(\frac{1+11K^3}{1-K^3}\right) \approx 0.843718586$$

$$Where, \quad packing\ constant,\ K = \frac{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}{4} \approx 0.114420648$$

It is to be noted that the value of the packing ratio $(r_p)_{\infty}$ is the maximum possible value which shows that approximate 84.37 % of the volume of any regular icosahedron can be packed by snugly fitting infinite no. of the spheres in each of its 12 identical vertices including the (volume of) largest inscribed sphere touching all 20 triangular faces.

Deduction: The above generalised formulae are applicable to locate any sphere with a radius R resting in a vertex (corner) at which n no. of edges meet together at angle α between any two consecutive of them such as the vertex of platonic solids, any of two identical & diagonally opposite vertices of uniform polyhedrons with congruent right kite faces & the vertex of right pyramid with regular n-gonal base. Hence for given values of $\boldsymbol{n}$ no. of edges meeting at a vertex, an angle $\boldsymbol{\alpha}$ between any two consecutive edges meeting at the vertex & radius $\boldsymbol{R}$ of the sphere resting in the vertex, all the important parameters are calculated as tabulated below

1.	**Distance of the centre of the sphere from the vertex of polyhedron**	$\frac{R\tan\frac{\pi}{n}}{\tan\frac{\alpha}{2}}$
2.	**Minimum distance of the sphere from the vertex of polyhedron**	$\frac{R\left(\tan\frac{\pi}{n}-\tan\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}}$
3.	**Distance of the centre of the sphere from each of edges meeting at the vertex of polyhedron**	$\frac{R\cos\frac{\alpha}{2}}{\cos\frac{\pi}{n}}$
4.	**Minimum distance of the sphere from each of edges meeting at the vertex of polyhedron**	$\frac{R\left(\cos\frac{\alpha}{2}-\cos\frac{\pi}{n}\right)}{\cos\frac{\pi}{n}}$
5.	**Normal height (depth) through which the vertex is truncated to best fit the sphere**	$\frac{R\left(\tan\frac{\pi}{n}-\tan\frac{\alpha}{2}\right)}{\tan\frac{\alpha}{2}}$
6.	**Edge length through which the vertex is truncated to best fit the sphere**	$R\tan\frac{\pi}{n}\ cosec\frac{\alpha}{2}\sqrt{\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}}$
7.	**Radius through which each face is filleted to best fit the sphere**	$R\tan\frac{\pi}{n}\tan\left(\frac{\pi+\alpha}{4}\right)\sqrt{\frac{\sin\left(\frac{\pi}{n}-\frac{\alpha}{2}\right)}{\sin\left(\frac{\pi}{n}+\frac{\alpha}{2}\right)}}$

$$\forall\ \alpha < \frac{2\pi}{n}\ \ \&\ n \geq 3$$

These generalised formulae are applicable **to locate any sphere with a radius R resting in a vertex (corner)** at which n no. of edges meet together at angle α between any two consecutive of edges. The resting sphere touches all n no. of faces meeting at that vertex but the sphere does not touch any of n no. of edges meeting at that vertex. Thus there is an equal minimum gap between sphere & each of the edges.

Let a be the **edge length** & N be the **no. of spheres snugly fitted/packed in each of the identical vertices** of the corresponding platonic solid then the important parameters are determined as tabulated below

Corresponding platonic solid	**Radius of N^{th} sphere in each vertex (excluding the largest one)**	**Total volume packed by all the spheres snugly fitted/packed in all the vertices of the corresponding platonic solid including the volume of the largest inscribed sphere**	**Packing ratio (ratio of the volume packed by all snugly fitted sphere to the volume of corresponding platonic solid)**	**% of max. packed volume**
Regular tetrahedron	$\frac{a}{2\sqrt{6}}\left(\frac{1}{2}\right)^N$	$\frac{\pi a^3}{36\sqrt{6}}\left\{1+\frac{4}{7}\left(\frac{2^{3N}-1}{2^{3N}}\right)\right\}$	$\frac{\pi}{6\sqrt{3}}\left\{1+\frac{4}{7}\left(\frac{2^{3N}-1}{2^{3N}}\right)\right\}$	$47.5\,\%$
Regular hexahedron (cube)	$\frac{a}{2}\left(2-\sqrt{3}\right)^N$	$\frac{\pi a^3}{6}\left\{1+\frac{4}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$	$\frac{\pi}{6}\left\{1+\frac{4}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$	$60.58\,\%$
Regular octahedron	$\frac{a}{\sqrt{6}}\left(2-\sqrt{3}\right)^N$	$\frac{\pi a^3}{9}\sqrt{\frac{2}{3}}\left\{1+\frac{3}{5}\left(3\sqrt{3}-5\right)\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$	$\frac{\pi}{3\sqrt{3}}\left\{1+\frac{3\left(3\sqrt{3}-5\right)}{5}\left(1-\left(2-\sqrt{3}\right)^{3N}\right)\right\}$	$67.57\,\%$
Regular dodecahedron	$\frac{a}{2}K^N\sqrt{\frac{25+11\sqrt{5}}{10}}$	$\frac{\pi a^3\sqrt{6100+2728\sqrt{5}}}{60}\left\{1+20K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$	$\frac{\pi}{15}\sqrt{\frac{65+29\sqrt{5}}{10}}\left\{1+20K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$	$77.73\,\%$
Regular icosahedron	$\frac{aK^N\left(3+\sqrt{5}\right)}{4\sqrt{3}}$	$\frac{\pi a^3\left(9+4\sqrt{5}\right)}{18\sqrt{3}}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$	$\frac{\pi\left(7+3\sqrt{5}\right)}{30\sqrt{3}}\left\{1+12K^3\left(\frac{1-K^{3N}}{1-K^3}\right)\right\}$	$84.37\,\%$

$$Where,\qquad packing\ constant,\ \ K = \frac{11+3\sqrt{5}-\sqrt{150+66\sqrt{5}}}{4} \approx 0.114420648$$

Note: *Above articles had been derived & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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Courtesy: *Advanced Geometry by Harish Chandra Rajpoot*