HCR's Inequality for three positive real numbers (Feb, 2015)

This is a conditional inequality, derived from a general formula for three externally touching circles, which always holds true for any three positive real numbers under certain conditions.

Let there be any three externally touching circles of radii a, b & c. Then the radius R of the circle circumscribing & internally touching three given circles, is given by the following formula

$$\boldsymbol{R = \frac{abc}{2\sqrt{abc(a+b+c)} - ab - bc - ca}}$$

1. The circumscribed circle will exist for three given radii a, b & c (where, $a \geq b \geq c > 0$) only if the following inequality is satisfied

$$c > \frac{ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}$$

The radii of three externally touching circles are always positive i.e. $a \geq b \geq c > 0 \quad \Rightarrow abc > 0$

The radius of circumscribed circle R will be positive (i.e. $R > 0$) if & only if the denominator of above fraction is positive hence

$$2\sqrt{abc(a+b+c)} - ab - bc - ca > 0$$

$$2\sqrt{abc(a+b+c)} > ab + bc + ca$$

Since both the sides are positive hence taking squares on both the sides (inequality is unchanged),

$$\left(2\sqrt{abc(a+b+c)}\right)^2 > (ab + bc + ca)^2$$

$$4abc(a+b+c) > a^2b^2 + b^2c^2 + c^2a^2 + 2a^2bc + 2ab^2c + 2abc^2$$

$$4abc(a+b+c) > a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a+b+c)$$

$$4abc(a+b+c) - 2abc(a+b+c) > a^2b^2 + b^2c^2 + c^2a^2$$

$$2abc(a+b+c) > a^2b^2 + b^2c^2 + c^2a^2$$

2. If radius c of the smallest circle is such that

$$c = \frac{ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}$$

then the radius of circumscribed circle is infinite i.e. $R = \infty$ & the following equality holds good

$$2abc(a+b+c) = a^2b^2 + b^2c^2 + c^2a^2$$

3. If radius c of the smallest circle is such that

$$c < \frac{ab}{\left(\sqrt{a} + \sqrt{b}\right)^2}$$

then the radius of circumscribed circle is negative i.e. $R < 0$ & the following inequality holds good

$$2abc(a+b+c) < a^2b^2 + b^2c^2 + c^2a^2$$

Since radii of all three externally touching circles are positive hence representing three radii by three positive real numbers there are only three possible cases out of which one is always satisfied.

Any three positive real numbers $\boldsymbol{a}, \boldsymbol{b}$ & $\boldsymbol{c}$ such that $\boldsymbol{a \geq b \geq c > 0}$ always satisfy one of **three possible cases** as follows

Case 1: If

$$c < \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$$

then the following inequality always holds good

$$\boldsymbol{2abc(a+b+c) < a^2b^2 + b^2c^2 + c^2a^2}$$

Case 2: If

$$c = \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$$

then the following equality always holds good

$$\boldsymbol{2abc(a+b+c) = a^2b^2 + b^2c^2 + c^2a^2}$$

Case 3: If

$$c > \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}$$

then the following inequality always holds good

$$\boldsymbol{2abc(a+b+c) > a^2b^2 + b^2c^2 + c^2a^2}$$

Thus we find that any three positive real numbers always satisfy one of the above conditional inequalities.