Application of HCR′s Inverse Cosine Formula

Minimum distance between any two points on a sphere

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We know that the great circle is a circle whose plane passes through the centre of sphere. The part of a great circle on a sphere is known as a great circle arc. The length of minor great circle arc (i.e. less than half great circle) joining any two arbitrary points on a sphere of finite radius is the minimum distance between those points. Here we are interested in finding out the minimum distance or great circle distance between any two arbitrary points on a spherical surface of finite radius (like globe) for the given values of latitudes & longitudes.

Let there be any two arbitrary points $\boldsymbol{A}(\boldsymbol{\phi_1}, \boldsymbol{\lambda_1})$ & $\boldsymbol{B}(\boldsymbol{\phi_2}, \boldsymbol{\lambda_2})$ lying on the surface of sphere of radius $\boldsymbol{R}$ & centre at the point O. The **angles of latitude** $\boldsymbol{\phi_1}$ & $\boldsymbol{\phi_2}$ are measured from the equator plane & the **angles of longitude** $\boldsymbol{\lambda_1}$ & $\boldsymbol{\lambda_2}$ are measured from a reference plane OPQ in the anticlockwise direction. Here, we are to find out the length of **great circle arc AB** joining the given points A & B. Draw the great circle arcs AD and BC passing through the points A & B respectively which intersect each other at the peak (pole) point P & intersect the equatorial line orthogonally (at 90^o) at the points D & C respectively. (As shown by the dashed arcs PD & PC in the figure 1)

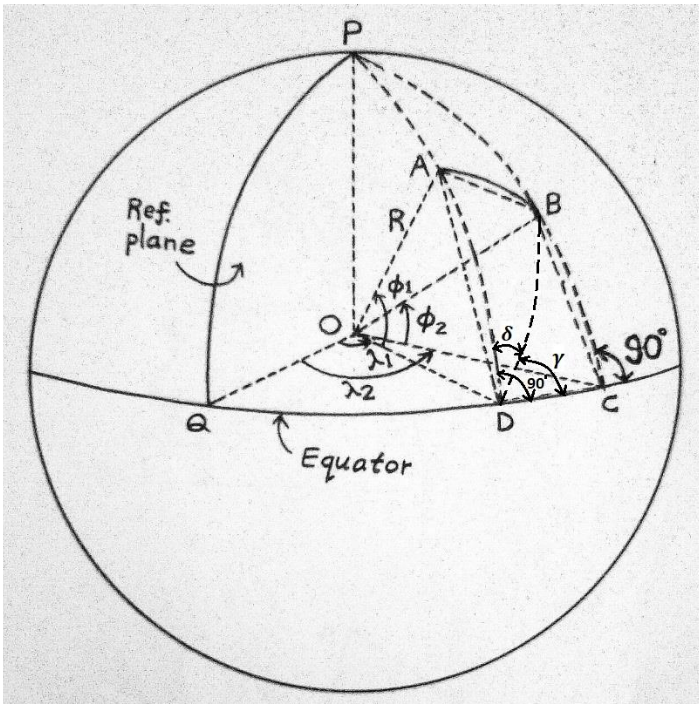

Figure 1: The dashed great circle arcs PD and PC passing through two given points A & B, intersecting each other at the peak (pole) point P, meet the equator orthogonally at the points D & C respectively on a spherical surface of finite radius *R*.

Join the points A, B, C & D by the dashed straight lines through the interior of sphere to get a **plane quadrilateral ABCD** and great circle arc BD.

Now, the angle between the orthogonal great circle arcs BC & CD, subtending the angles $\alpha = \phi_2$ & $\beta = \lambda_2 - \lambda_1$ respectively at the centre O of the sphere, meeting each other at the common end point C, is $\theta = {}^{\pi}/_{2}$. Now, consider the tetrahedron OBCD formed by joining the points B, C and D to the centre O (see the fig-1). A diahedral angle is the angle between two intersecting planes measured in a plane perpendicular to the both the intersecting planes. Now, the diahedral angle say θ between the lateral triangular faces BOC and COD is given by **HCR's Inverse Cosine Formula** [1] as follows

$$\cos\theta = \frac{\cos\alpha' - \cos\alpha\cos\beta}{\sin\alpha\sin\beta} \qquad \dots\dots\dots\dots\dots\dots (1)$$

where, is θ is the diahedral angle between lateral triangular faces BOC and COD intersecting each other at the line OC which is equal to the angle between great circle arcs BC and CD intersecting each other perpendicularly i.e. $\theta = \frac{\pi}{2}$ which is opposite to $\alpha' = \angle BOD$ and

$\alpha' = \angle BOD,\ \alpha = \angle BOC,\ \beta = \angle COD$ are the vertex angles of triangular faces BOD, BOC and COD respectively meeting at the apex O of tetrahedron OBCD and angle $\alpha'$is always opposite to the diahedral angle θ (as shown in the above fig-1 and fig-2 below).

Now, substituting the corresponding values i.e. $\theta = \frac{\pi}{2}$, $\alpha' = \angle BOD$, $\alpha = \angle BOC = \phi_2$ and $\beta = \angle COD = \lambda_2 - \lambda_1$ in the above Eq(1) (HCR's Inverse Cosine Formula) as follows

$$\cos\frac{\pi}{2} = \frac{\cos\angle BOD - \cos\phi_2\cos(\lambda_2 - \lambda_1)}{\sin\phi_2\sin(\lambda_2 - \lambda_1)}$$

$$\cos\angle BOD - \cos\phi_2\cos(\lambda_2 - \lambda_1) = 0$$

$$\cos\angle BOD = \cos\phi_2\cos(\lambda_2 - \lambda_1) \qquad \dots\dots\dots\dots\dots(2)$$

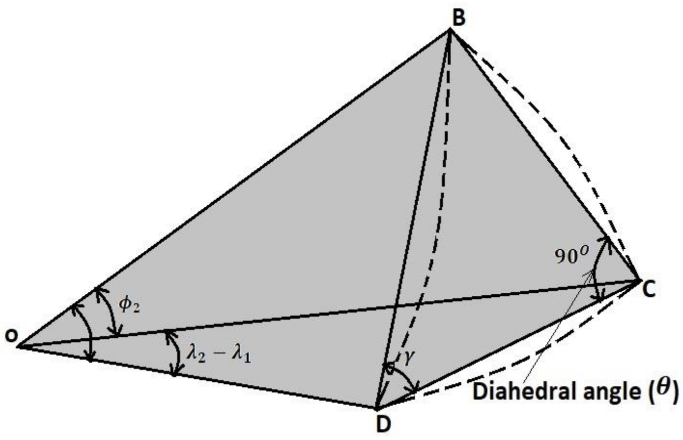

Figure 2: The diahedral angle between the triangular faces BOC and COD of tetrahedron OBCD is equal to the angle between great arcs BC and CD which intersect each other at right angle because the great arc BC intersects equatorial line (i.e. great arc CD) at right angle.

Similarly, from the figure-2, the diahedral angle say γ between the lateral triangular faces BOD and COD of tetrahedron OBCD is obtained by substituting the corresponding values i.e. $\theta = \gamma$ which is opposite to $\alpha' = \angle BOC = \phi_2$, $\alpha = \angle BOD$ and $\beta = \angle COD = \lambda_2 - \lambda_1$ in the above Eq(1) (HCR's Inverse Cosine Formula) as follows

$$\cos\gamma = \frac{\cos\phi_2 - \cos\angle BOD\cos(\lambda_2 - \lambda_1)}{\sin\angle BOD\sin(\lambda_2 - \lambda_1)}$$

$$= \frac{\cos\phi_2 - \cos\phi_2\cos(\lambda_2 - \lambda_1)\cos(\lambda_2 - \lambda_1)}{\sin\angle BOD\sin(\lambda_2 - \lambda_1)} \qquad \text{(Setting value of } \cos\angle BOD \text{ from Eq(2))}$$

$$= \frac{\cos\phi_2\,(1 - \cos^2(\lambda_2 - \lambda_1))}{\sin\angle BOD\sin(\lambda_2 - \lambda_1)} = \frac{\cos\phi_2\sin^2(\lambda_2 - \lambda_1)}{\sin\angle BOD\sin(\lambda_2 - \lambda_1)} = \frac{\cos\phi_2\sin(\lambda_2 - \lambda_1)}{\sin\angle BOD}$$

$$\cos\gamma = \frac{\cos\phi_2\sin(\lambda_2 - \lambda_1)}{\sin\angle BOD} \qquad \dots\dots\dots\dots\dots(3)$$

Now, from the figure-3, consider the tetrahedron OABD formed by joining the points A, B and D to the centre O (also shown in the above fig-1). Now, the diahedral angle say δ between the lateral triangular faces AOD and BOD is obtained by substituting the corresponding values i.e. $\theta = \delta$ which is opposite to $\alpha' = \angle AOB$, $\alpha = \angle AOD = \phi_1$ and $\beta = \angle BOD$ in the above Eq(1) (HCR's Inverse Cosine Formula) as follows

$$\cos\delta = \frac{\cos\angle AOB - \cos\phi_1\cos\angle BOD}{\sin\phi_1\sin\angle BOD}$$

$$\cos(90^o - \gamma) = \frac{\cos\angle AOB - \cos\phi_1\cos\angle BOD}{\sin\phi_1\sin\angle BOD}$$

$$(\because \gamma + \delta = 90^o)$$

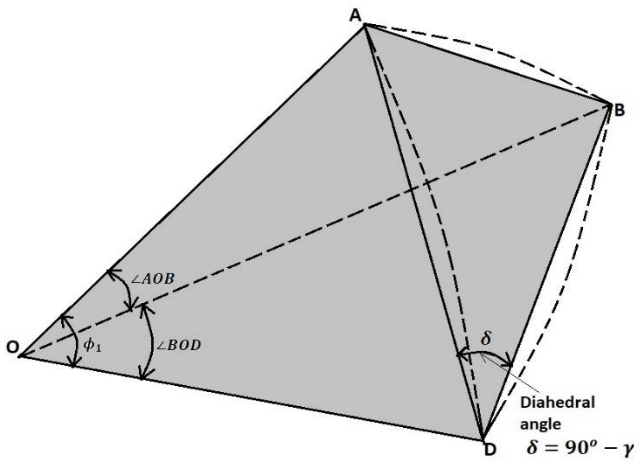

Figure 3: The diahedral angle δ between the triangular faces AOD and BOD is equal to the angle between great arcs AD and BD which is opposite to the face angle $\angle AOB$ of tetrahedron OABD.

$$\sin\gamma = \frac{\cos\angle AOB - \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1)}{\sin\phi_1\sin\angle BOD} \qquad \text{(Setting value of } \cos\angle BOD \text{ from Eq(2))}$$

$$\sin\phi_1\sin\gamma\sin\angle BOD = \cos\angle AOB - \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1)$$

$$\cos\angle AOB = \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\gamma\sin\angle BOD$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\angle BOD\sqrt{\sin^2\gamma} \qquad (\because 0\le\gamma\le\pi)$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\angle BOD\sqrt{1-\cos^2\gamma}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\angle BOD\sqrt{1-\left(\frac{\cos\phi_2\sin(\lambda_2-\lambda_1)}{\sin\angle BOD}\right)^2} \qquad \text{(from Eq(3))}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\angle BOD\sqrt{\frac{\sin^2\angle BOD - \cos^2\phi_2\sin^2(\lambda_2-\lambda_1)}{\sin^2\angle BOD}}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sqrt{1-\cos^2\angle BOD - \cos^2\phi_2\sin^2(\lambda_2-\lambda_1)}$$

Substituting the value of $\cos\angle BOD$ from the above Eq(2),

$$\cos\angle AOB = \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sqrt{1-(\cos\phi_2\cos(\lambda_2-\lambda_1))^2 - \cos^2\phi_2\sin^2(\lambda_2-\lambda_1)}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sqrt{1-\cos^2\phi_2\cos^2(\lambda_2-\lambda_1) - \cos^2\phi_2\sin^2(\lambda_2-\lambda_1)}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sqrt{1-\cos^2\phi_2\left(\sin^2(\lambda_2-\lambda_1) + \cos^2(\lambda_2-\lambda_1)\right)}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sqrt{1-\cos^2\phi_2\,(1)}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sqrt{\sin^2\phi_2}$$

$$= \cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\phi_2 \qquad (\because 0\le\phi_2\le\pi)$$

$$\Rightarrow \boldsymbol{\angle AOB} = \cos^{-1}(\cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\phi_2) \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

The above $\angle AOB$ is the angle subtended at the centre O by the great circle arc AB joining the given points A and B lying on the sphere (as shown in the above fig-1). Therefore the minimum distance between two points A and B on the sphere will be equal to the great circle arc AB given as follows

$$\text{The length of great circle arc AB} = \text{Central angle} \times \text{Radius of sphere} = \boldsymbol{\angle AOB \times R}$$

$$= \cos^{-1}(\cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\phi_2) \times R$$

$$= R\cos^{-1}(\cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\phi_2)$$

$\therefore$ Minimum distance between the points $A(\phi_1, \lambda_1)$ & $B(\phi_2, \lambda_2)$ lying on sphere of radius R,

$$\mathbf{d_{min}} = \boldsymbol{R\cos^{-1}(\cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1) + \sin\phi_1\sin\phi_2)}$$

$$\forall \quad \boldsymbol{0 \le \phi_1, \phi_2, |\lambda_2-\lambda_1| \le \pi}$$

The above formula is called **Great-circle distance formula** which gives the minimum distance between any two arbitrary points lying on the sphere for given latitudes and longitudes.

NOTE: It's worth noticing that the above formula of **Great-circle distance** has symmetry i.e. if ϕ_1 & ϕ_2 and λ_1 & λ_2 are interchanged, the formula remains unchanged. It also implies that if the locations of two points for given values of latitude & longitude are interchanged, the distance between them does not change at all.

Since the equator plane divides the sphere into two equal hemispheres hence the above formula is applicable to find out the minimum distance between any two arbitrary points lying on any of two hemispheres. So for the convenience, the equator plane of the sphere should be taken in such a way that the given points lie on one of the two hemispheres resulting from division of sphere by the reference equator plane.

Since the maximum value of $\cos^{-1}(x)$ is π hence the max. of min. distance between two points on a sphere is

$$= R(\pi) = \boldsymbol{\pi R} = \textbf{half of the perimeter of a great circle passing through given points}$$

Case 1: If both the given points lie on the equator of the sphere then substituting $\phi_1 = \phi_2 = 0$ in the great-circle distance formula, we obtain

$$\mathbf{d_{min}} = R\cos^{-1}(\cos 0 \cos 0 \cos(\lambda_2 - \lambda_1) + \sin 0 \sin 0) = R\cos^{-1}(\cos(\lambda_2 - \lambda_1)) = R|\lambda_2 - \lambda_1| = \boldsymbol{R|\Delta\lambda|}$$

The above result shows that the minimum distance between any two points lying on the equator of the sphere depends only on the difference of longitudes of two given points & the radius of the sphere. In this case, the minimum distance between such two points is simply the product of radius R and the angle $\Delta\lambda$ between them measured on the equatorial plane of sphere. This

If both the given points lie diametrically opposite on the equator of the sphere then substituting $|\boldsymbol{\Delta\lambda}| = \boldsymbol{\pi}$ in above expression, the minimum distance between such points

$$R|\Delta\lambda| = R(\pi) = \boldsymbol{\pi R} = \text{half of the perimeter of a great circle passing through thegiven points}$$

Case 2: If both the given points lie on a great circle arc normal to the equator of the sphere then substituting $\lambda_2 - \lambda_1 = \Delta\lambda = 0$ in the formula of great-circle distance, we obtain

$$\mathbf{d_{min}} = R\cos^{-1}(\cos\phi_1 \cos\phi_2 \cos(0) + \sin\phi_1 \sin\phi_2)$$

$$= R\cos^{-1}(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) = R\cos^{-1}(\cos(\phi_1 - \phi_2)) = \boldsymbol{R|\phi_1 - \phi_2|}$$

In this case, the minimum distance between such two points is simply the product of radius R and the angle $|\phi_1 - \phi_2|$ between them measured on plane normal to the equatorial plane of sphere.

Case 3: If both the given points lie at the same latitude of the sphere then substituting $\phi_1 = \phi_2 = \phi$ in the formula of great-circle distance, we obtain

$$\mathbf{d_{min}} = R\cos^{-1}(\cos\phi \cos\phi \cos(\lambda_2 - \lambda_1) + \sin\phi \sin\phi) = \boldsymbol{R\cos^{-1}(\cos^2\phi \cos(\lambda_2 - \lambda_1) + \sin^2\phi)}$$

In this case, the minimum distance between such two points is dependent on the radius and both the latitude, and longitudes.

Illustrative Example

Consider any two arbitrary points A & B having respective angles of latitude $\boldsymbol{\phi_1 = 40^o}$ & $\boldsymbol{\phi_2 = 75^o}$ & the difference of angles of longitude $\boldsymbol{\Delta\lambda = \lambda_2 - \lambda_1 = 55^o}$ on a sphere of radius 25 cm**.** The minimum distance between given points lying on the sphere is obtained by substituting the corresponding values in the above **great-circle distance formula** as follows

$$\mathrm{d_{min}} = R\cos^{-1}(\cos\phi_1\cos\phi_2\cos(\lambda_2-\lambda_1)+\sin\phi_1\sin\phi_2)$$

$$= 25\cos^{-1}(\cos 40^o\cos 75^o\cos(55^o)+\sin 40^o\sin 75^o)$$

$$= 25\cos^{-1}(0.7346063699582707) \approx 18.64274952833712\ cm$$

The above result also shows that the points A & B divide the perimeter $= 2\pi(25) \approx 157.07963267948966 cm$ of the great circle in two great circles arcs (one is minor arc AB of length $\approx 18.64274952833712\ cm$ & other is major arc AB of length $\approx 138.43688315115253\ cm$) into a ratio $\approx {18.64274952833712}/{138.43688315115253} \approx \mathbf{1:7.4}$

Conclusion: It can be concluded that the analytic formula of great-circle distance derived here directly gives the correct values of the great-circle distance between any two arbitrary points on the sphere because there is no approximation in the formula. This is extremely useful formula to compute the minimum distance between any two arbitrary points lying on a sphere of finite radius which is equally applicable in global positioning system. This formula is the most general formula to calculate the geographical distance between any two points on the globe for the given latitudes & longitudes. This is a high precision formula which gives the correct values for all the distances on the tiny sphere as well as the large spheres such as Earth, and other giant planets assuming them the perfect spheres.

Note: Above articles had been derived *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

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[1]: H C Rajpoot. (2014). HCR's Inverse Cosine Formula, (Solution of internal & external angles of a tetrahedron). Academia.edu.

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