$\boldsymbol{Volume\ \&\ surface\ area\ of\ a\ right\ circular\ cone\ cut\ by\ a\ plane\ parallel\ to\ its\ symmetrical\ axis}$

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1. Introduction:

When a right circular cone is thoroughly cut by a plane parallel to its symmetrical (longitudinal) axis, then we get either a hyperbola or a pair of straight lines. Here we are interested to compute the volume & surface of right circular cone left after cutting by a plane parallel to its symmetrical axis by computing the volume & surface area of removed part of original cone. (See figure 1, showing a right circular cone ABC of vertical height H & base radius R being cut by a plane ST (dotted line) parallel to its symmetrical axis AO at a distance x). Thus we have three known values as follows

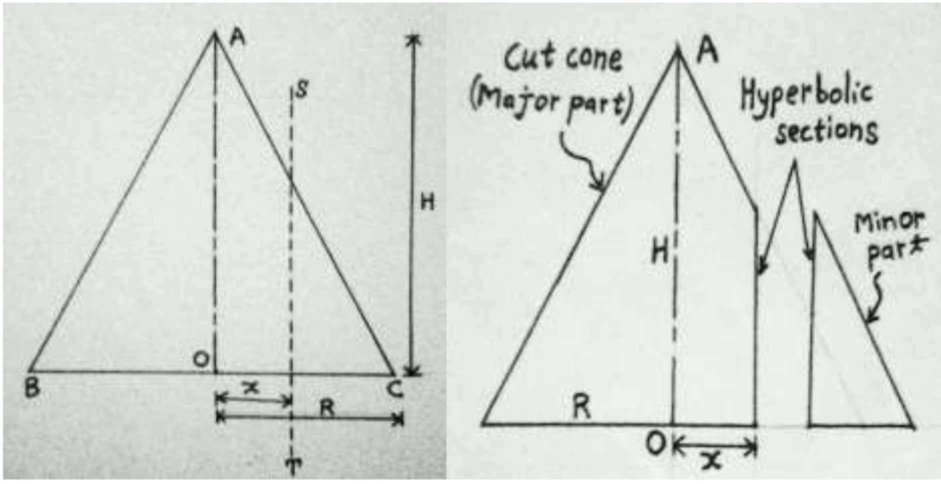

Figure 1: A right circular cone of vertical height H & base radius R is being cut by a parallel plane at a distance x from the symmetrical axis AO. Thus the original cone is divided into two parts, one is major & other is minor with hyperbolic plane sections

$\boldsymbol{H} =$ vertical height of right circular cone

$\boldsymbol{R} =$ base radius of right circular cone

$\boldsymbol{x} =$ distance of cutting plane from the symmetrical(longitudinal) axis of right circular cone

We have to derive the mathematical expressions of volume & surface area of cut cone (i.e. major part left over after removing smaller/minor part) in terms of above three known/given values.

2. Derivation of volume of cut cone (Major part):

Consider a right circular cone ABC of vertical height H & base radius R cut by a plane parallel to its symmetrical axis AO at a distance x. In order to derive the expression of volume of the cut cone left over after removing smaller/minor part, it is easier to derive the expression of the volume of removed part from original right circular cone. Consider the circular base of cone on the XY-plane & symmetrical axis AO is coincident with Z-axis. Now consider any arbitrary point say $P(r, \theta, z)$ on curved surface of minor part to be removed from cone (see left image in fig-2)

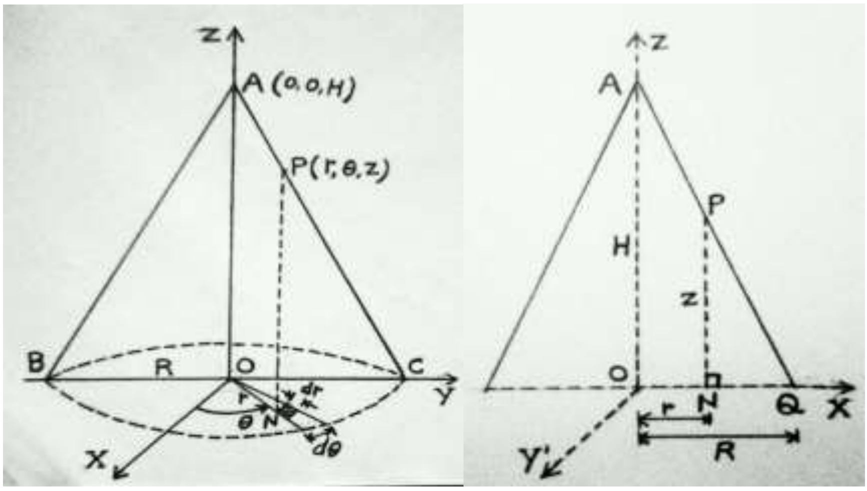

Figure 2: A parametric point $P(r, \theta, z)$ lies on the curved surface of minor part of original right circular cone. A perpendicular PN is dropped from the point P to the XY-plane within the region of minor part to be removed from original cone

Drop a perpendicular PN from point P to the XY-plane within region to be removed. Now, in similar right triangles ΔAOQ & ΔPNQ (see right image in fig-2)

$$\frac{PN}{NQ} = \frac{AO}{OQ} \;\Rightarrow\; \frac{z}{R-r} = \frac{H}{R} \quad \Rightarrow \quad \boldsymbol{z = \frac{H}{R}(R-r)}$$

Now, consider an **infinitely small rectangular area, $dA = (rd\theta)dr$ on the XY-plane within the region of minor part to be removed from original right circular cone** (see figure-3 below) & let it extend vertically upward to a height z up to the point P (as shown by dotted line PN in the figure-2 above)

Now, the volume of the elementary cylinder of normal height z & cross sectional area $(rd\theta)dr$

$$dV = z(rd\theta)dr = zrdrd\theta$$

By integrating above volume dV, the total volume of removed part (i.e. minor part) of cone

$$Volume\ of\ removed\ part\ (minor\ part) = \int dV = \iint zrdrd\theta$$

The minor part to be removed from original cone is symmetrical about XZ-plane & its base area is symmetrical about x-axis (As shown by figure-3) hence using symmetry of solid minor part to be removed from original cone & applying the proper limits, the volume of removed minor part

$$V_{minor} = \iint zrdrd\theta = 2\int_0^{\angle COM}\int_{xsec\theta}^{R} zrdrd\theta$$

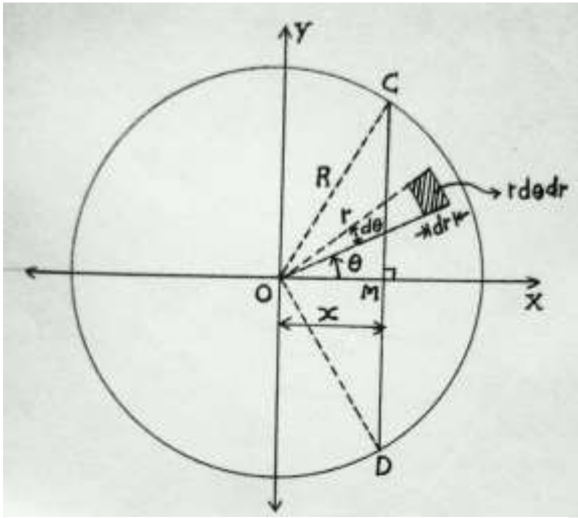

Figure 3: It is the top view of circular base of right cone which is cut by a plane CD (normal to plane of paper). The minor part (to the left of plane CD) of original right circular cone is symmetrical about XZ-plane.

Substituting the value of z, we get

$$V_{minor} = 2\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \int_{xsec\theta}^{R} \frac{H}{R}(R-r)rdr$$

$$= \frac{2H}{R}\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \int_{xsec\theta}^{R} (R-r)rdr$$

$$= \frac{2H}{R}\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \left[\frac{Rr^2}{2} - \frac{r^3}{3}\right]_{xsec\theta}^{R}$$

$$= \frac{2H}{R}\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} \left(\frac{R^3 - 3Rx^2sec^2\theta + 2x^3sec^3\theta}{6}\right) d\theta$$

$$= \frac{H}{3R}\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} (R^3 - 3Rx^2sec^2\theta + 2x^3sec^3\theta)\, d\theta \qquad (\text{Note: } \boldsymbol{x} \text{ is an } \textbf{arbitrary constant})$$

$$= \frac{H}{3R}\left[R^3\theta - 3Rx^2tan\theta + x^3(sec\theta tan\theta + \ln|sec\theta + tan\theta|)\right]_0^{\cos^{-1}\left(\frac{x}{R}\right)}$$

$$= \frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right) - 3Rx^2tan\left(\cos^{-1}\left(\frac{x}{R}\right)\right) + x^3\left(sec\left(\cos^{-1}\left(\frac{x}{R}\right)\right)tan\left(\cos^{-1}\left(\frac{x}{R}\right)\right) + \ln\left|sec\left(\cos^{-1}\left(\frac{x}{R}\right)\right) + tan\left(\cos^{-1}\left(\frac{x}{R}\right)\right)\right|\right)\right]$$

$$= \frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right) - 3Rx^2\frac{\sqrt{R^2-x^2}}{x} + x^3\left(\frac{R}{x}\frac{\sqrt{R^2-x^2}}{x} + \ln\left|\frac{R}{x} + \frac{\sqrt{R^2-x^2}}{x}\right|\right)\right]$$

$$= \frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right) - 3Rx\sqrt{R^2-x^2} + Rx\sqrt{R^2-x^2} + x^3\ln\left|\frac{R+\sqrt{R^2-x^2}}{x}\right|\right]$$

$$=\frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right)-2Rx\sqrt{R^2-x^2}+x^3\ln\left(\frac{R+\sqrt{R^2-x^2}}{x}\right)\right]$$

Hence, the **volume** $(\boldsymbol{V_{minor}})$ **of the minor part removed from original right circular cone**,

$$\boldsymbol{V_{minor}=\frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right)-2Rx\sqrt{R^2-x^2}+x^3\ln\left(\frac{R+\sqrt{R^2-x^2}}{x}\right)\right]\quad\forall\ 0\le x\le R}$$

Hence, the **volume** $(\boldsymbol{V_{major}})$ **of cut right circular cone (i.e. major part left over from original cone),**

$$V_{major}=\text{total volume of original right circular cone}-\text{volume of removed part}$$

$$=\frac{1}{3}\pi R^2H-V_{minor}$$

$$\boldsymbol{V_{major}=\frac{1}{3}\pi R^2H-\frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right)-2Rx\sqrt{R^2-x^2}+x^3\ln\left(\frac{R+\sqrt{R^2-x^2}}{x}\right)\right]}$$

Important deductions: 1. If the cutting plane passes through the symmetrical (longitudinal) axis of the original right circular cone then substituting $x=0$ in the above expressions, we get

$$V_{minor}=\lim_{x\to 0}\frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right)-2Rx\sqrt{R^2-x^2}+x^3\ln\left(\frac{R+\sqrt{R^2-x^2}}{x}\right)\right]=\frac{1}{6}\pi R^2H$$

$$=\frac{1}{2}(\text{volume of original cone})$$

$$V_{major}=\frac{1}{3}\pi R^2H-V_{minor}=\frac{1}{3}\pi R^2H-\frac{1}{6}\pi R^2H=\frac{1}{6}\pi R^2H=\frac{1}{2}(\text{volume of original cone})$$

The above deduction is obviously true that a plane passing through the symmetrical (longitudinal) axis divides the right circular cone into two equal parts each having a volume half that of the original cone.

2. Substituting $x=R$ in the above expressions, we get

$$V_{minor}=\lim_{x\to R}\frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right)-2Rx\sqrt{R^2-x^2}+x^3\ln\left(\frac{R+\sqrt{R^2-x^2}}{x}\right)\right]=0\ \&\ V_{major}=\frac{1}{3}\pi R^2H$$

It is obvious that the cutting plane, parallel to the symmetrical axis & tangent to the base circle of a right circular cone doesn't remove any part of cone.

3. Derivation of area of the curved surface of cut cone (Major part):

In order to derive the expression of area of curved surface of the cut cone (major part) left over after removing smaller/minor part, it is easier to derive the expression of area of curved surface of minor part to be removed from original right circular cone. Consider the circular base of original cone on the XY-plane & symmetrical axis AO is coincident with z-axis. Now, consider any arbitrary point say $P(r,\theta,z)$ on the curved surface of minor part to be removed from the original right circular cone (As shown in the figure-4)

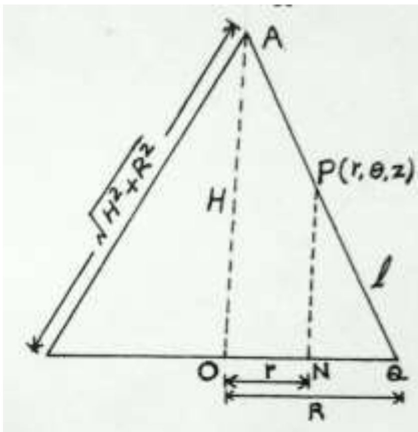

Figure 4: A parametric point $\boldsymbol{P(r,\theta,z)}$ lies on the curved surface of minor part of original right circular cone. $\boldsymbol{PQ=l}$ is the slant length of minor part

Now, in similar right triangles ΔAOQ & ΔPNQ (see fig-3)

$$\frac{PQ}{AQ}=\frac{NQ}{OQ} \Rightarrow \frac{l}{\sqrt{H^2+R^2}}=\frac{R-r}{R} \quad \Rightarrow \quad l=\frac{(R-r)\sqrt{H^2+R^2}}{R}$$

On differentiating slant length l w.r.t. r, we get

$$\frac{dl}{dr}=-\frac{\sqrt{H^2+R^2}}{R} \quad \Rightarrow \quad \boldsymbol{dl=-\frac{\sqrt{H^2+R^2}}{R}dr}$$

Now, consider an **infinitely small rectangular area, $\boldsymbol{dS=(rd\theta)dl}$ on the curved surface of minor part to be removed from original right circular cone.** Hence by integrating the elementary area, the total area of curved surface of minor part to be removed

$$S_{minor}=\int dS=\iint rd\theta dl$$

The curved surface of minor part to be removed from original cone is symmetrical about XZ-plane & its base area is symmetrical about x-axis (As shown by figure-3 above) hence using symmetry of curved surface to be removed from original cone & applying the proper limits, area of curved surface of minor part to be removed

$$S_{minor}=2\int_{0}^{\angle COM}\int_{R}^{x} rd\theta dl$$

Substituting the value of dl, we get

$$S_{minor}=2\int_{0}^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \int_{R}^{x} r\left(-\frac{\sqrt{H^2+R^2}}{R}dr\right)$$

$$=\frac{2\sqrt{H^2+R^2}}{R}\int_{0}^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \int_{x}^{R} rdr$$

$$=\frac{2\sqrt{H^2+R^2}}{R}\int_{0}^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \left[\frac{r^2}{2}\right]_{x}^{R}$$

$$=\frac{2\sqrt{H^2+R^2}}{R}\int_{0}^{\cos^{-1}\left(\frac{x}{R}\right)} \left(\frac{R^2-x^2}{2}\right)d\theta$$

$$=\frac{2\sqrt{H^2+R^2}}{R}\left(\frac{R^2-x^2}{2}\right)\int_{0}^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \quad (\text{Note: } \boldsymbol{x} \text{ is an } \textbf{arbitrary constant})$$

$$=\left(\frac{R^2-x^2}{R}\right)\sqrt{H^2+R^2}\,[\theta]_{0}^{\cos^{-1}\left(\frac{x}{R}\right)}$$

$$=\left(\frac{R^2-x^2}{R}\right)\sqrt{H^2+R^2}\cos^{-1}\left(\frac{x}{R}\right)$$

Hence, the **area of curved surface ($\boldsymbol{S_{minor}}$) of the minor part removed from original right circular cone**,

$$\boldsymbol{S_{minor}=\frac{(R^2-x^2)\sqrt{H^2+R^2}}{R}\cos^{-1}\left(\frac{x}{R}\right) \qquad \forall\ 0\le x\le R}$$

Hence, the **area of curved surface $\left(\boldsymbol{S_{major}}\right)$ of cut right circular cone (i.e. major part left over from original cone),**

$$S_{major} = \text{total area of cuved surface of original right circular cone} \\ - \text{ area of curved surface of removed part}$$

$$= \pi R\sqrt{H^2 + R^2} - S_{minor}$$

$$\boldsymbol{S_{major} = \pi R\sqrt{H^2 + R^2} - \frac{(R^2 - x^2)\sqrt{H^2 + R^2}}{R}\cos^{-1}\left(\frac{x}{R}\right)}$$

Important deductions: 1. If the cutting plane passes through the symmetrical (longitudinal) axis of the original right circular cone then substituting $x = 0$ in the above expressions, we get

$$S_{minor} = \lim_{x\to 0}\frac{(R^2 - x^2)\sqrt{H^2 + R^2}}{R}\cos^{-1}\left(\frac{x}{R}\right) = \frac{1}{2}\pi R\sqrt{H^2 + R^2} = \frac{1}{2}(\text{area of curved surface of original cone})$$

$$S_{major} = \pi R\sqrt{H^2 + R^2} - S_{minor} = \pi R\sqrt{H^2 + R^2} - \frac{1}{2}\pi R\sqrt{H^2 + R^2} = \frac{1}{2}\pi R\sqrt{H^2 + R^2}$$
$$= \frac{1}{2}(\text{area of curved surface of original cone})$$

The above deduction is obviously true that a plane passing through the symmetrical (longitudinal) axis divides the right circular cone into two equal parts each having area of curved surface half that of the original cone.

2. Substituting $x = R$ in the above expressions, we get

$$S_{minor} = \lim_{x\to R}\frac{(R^2 - x^2)\sqrt{H^2 + R^2}}{R}\cos^{-1}\left(\frac{x}{R}\right) = 0 \ \ \& \ \ S_{major} = \pi R\sqrt{H^2 + R^2}$$

It is obvious that the cutting plane, parallel to the symmetrical axis & tangent to the base circle of a right circular cone doesn't remove any part of cone.

4. Derivation of area of the hyperbolic section of cut cone (Major part):

The hyperbolic sections of both major & minor parts cut from the right circular cone are identical & equal in dimensions. In order to derive the expression of **area of hyperbolic section of the cut cone** (major part) or removed/minor part, configure the hyperbolic section on the XY-plane symmetrically about the x-axis such that $A(a, 0)$ is the vertex of right branch of hyperbola (As shown in the figure-5) Length of OA can be easily obtained by rule of similarity of right triangles, as follows

$$OA = \frac{Hx}{R} \quad (\textbf{Note}: \boldsymbol{x} \text{ is an } \textbf{arbitrary constant})$$

Points C, D, M, N lie on the base of the cut cone (major part). In right ΔCMO (see figure-3 above)

$$CD = 2CM = 2\sqrt{R^2 - x^2} \ \Rightarrow \ BC = \frac{CD}{2} = \sqrt{R^2 - x^2}$$

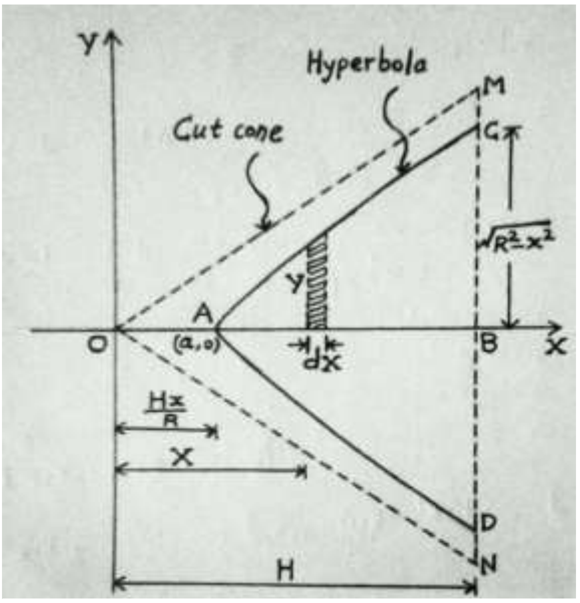

Figure 5: Hyperbolic section of the cut cone is configured symmetrically about the x-axis. $A(a, 0)$ is the vertex of hyperbola

Let the **standard equation of hyperbola** be as follows

$$\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$$

Since, $= OA = \frac{Hx}{R}$, & the point $C\left(H, \sqrt{R^2 - x^2}\right)$ lies on the hyperbola, hence substituting the corresponding values of X, Y & a in the equation of hyperbola,

$$\frac{H^2}{\left(\frac{Hx}{R}\right)^2}-\frac{\left(\sqrt{R^2-x^2}\right)^2}{b^2}=1 \;\Rightarrow\; \boldsymbol{b^2=x^2}$$

Now, substituting the values of semi-major & semi-minor axes a & b in the equation of hyperbola,

$$\boldsymbol{\frac{X^2}{\left(\frac{Hx}{R}\right)^2}-\frac{Y^2}{x^2}=1 \quad \Rightarrow \; Y=\frac{R}{H}\sqrt{X^2-\left(\frac{Hx}{R}\right)^2}}$$

Now, consider an elementary rectangular slab of area $dA=YdX$ & then integrating dA to get the total area of plane hyperbolic section of the cut cone as follows

$$A_{hyper}=\int dA=\int YdX=2\int_{\frac{Hx}{R}}^{H}\frac{R}{H}\sqrt{X^2-\left(\frac{Hx}{R}\right)^2}\,dX$$

$$=\frac{2R}{H}\int_{\frac{Hx}{R}}^{H}\sqrt{X^2-\left(\frac{Hx}{R}\right)^2}\,dX \qquad\qquad (\boldsymbol{Note}:\boldsymbol{x}\text{ is an }\boldsymbol{arbitrary\ constant})$$

$$=\frac{2R}{H}\left[\frac{1}{2}X\sqrt{X^2-\left(\frac{Hx}{R}\right)^2}-\frac{1}{2}\left(\frac{Hx}{R}\right)^2\ln\left|X+\sqrt{X^2-\left(\frac{Hx}{R}\right)^2}\right|\right]_{\frac{Hx}{R}}^{H}$$

$$=\frac{R}{H}\left[H\sqrt{H^2-\left(\frac{Hx}{R}\right)^2}-\left(\frac{Hx}{R}\right)^2\ln\left|H+\sqrt{H^2-\left(\frac{Hx}{R}\right)^2}\right|-0+\left(\frac{Hx}{R}\right)^2\ln\left|\frac{Hx}{R}\right|\right]$$

$$=\frac{R}{H}\left[\frac{H^2\sqrt{R^2-x^2}}{R}-\left(\frac{Hx}{R}\right)^2\ln\left(\frac{H\left(R+\sqrt{R^2-x^2}\right)}{R}\right)+\left(\frac{Hx}{R}\right)^2\ln\left(\frac{Hx}{R}\right)\right]$$

$$=H\sqrt{R^2-x^2}+\frac{Hx^2}{R}\ln\left(\frac{\frac{Hx}{R}}{\frac{H\left(R+\sqrt{R^2-x^2}\right)}{R}}\right)$$

$$=H\sqrt{R^2-x^2}+\frac{Hx^2}{R}\ln\left(\frac{x}{R+\sqrt{R^2-x^2}}\right)$$

Hence, the **area of hyperbolic section** $\left(A_{hyper}\right)$ **of the cut cone or removed/minor part**,

$$\boldsymbol{A_{hyper}=H\sqrt{R^2-x^2}+\frac{Hx^2}{R}\ln\left(\frac{x}{R+\sqrt{R^2-x^2}}\right) \qquad \forall\ 0\le x\le R}$$

Important deductions: 1. If the cutting plane passes through the symmetrical (longitudinal) axis of the original right circular cone then substituting $x=0$ in the above expressions, we get

$$A_{hyper}=\lim_{x\to 0}H\sqrt{R^2-x^2}+\frac{Hx^2}{R}\ln\left(\frac{x}{R+\sqrt{R^2-x^2}}\right)=HR$$
$$=\text{area of isosceles triangular section with base 2R \& altitude H}$$

The above deduction is obviously true that a plane passing through the symmetrical (longitudinal) axis divides the right circular cone into two equal parts each with an isosceles triangular section instead of hyperbolic section i.e. cutting plane passing through the symmetrical axis of a right circular cone gives a pair of straight lines instead of a conic i.e. hyperbola.

2. Substituting $x = R$ in the above expressions, we get

$$A_{hyper} = \lim_{x \to R} H\sqrt{R^2 - x^2} + \frac{Hx^2}{R} \ln\left(\frac{x}{R + \sqrt{R^2 - x^2}}\right) = 0$$

It is obvious that the cutting plane, parallel to the symmetrical axis & tangent to the base circle of a right circular cone doesn't remove any part of cone.

5. Derivation of area of plane base of cut cone (Major part):

In order to derive the expression of area of circular base of the cut cone (major part) left over after removing smaller/minor part, it is easier to derive the expression of area of circular base of minor part to be removed from original right circular cone. Configure the circular base of original cone on the XY-plane such that the base of minor part to be removed is symmetrical about x-axis (As shown in the figure-3 above)

Now, consider an infinitely small rectangular area, $dA = (rd\theta)dr$ on the XY-plane within the region of minor part to be removed from original right circular cone. Now, considering symmetry of circular base of minor part about x-axis & applying the proper limits, the area of circular base of removed minor part

$$(A_b)_{minor} = \iint rdrd\theta = 2\int_0^{\angle COM} \int_{xsec\theta}^{R} rdrd\theta = 2\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} \int_{xsec\theta}^{R} rdrd\theta$$

$$= 2\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} d\theta \left[\frac{r^2}{2}\right]_{xsec\theta}^{R} = 2\int_0^{\cos^{-1}\left(\frac{x}{R}\right)} \left(\frac{R^2 - x^2 sec^2\theta}{2}\right) d\theta$$

$$= \int_0^{\cos^{-1}\left(\frac{x}{R}\right)} (R^2 - x^2 sec^2\theta)\, d\theta \qquad \text{(Note: } \boldsymbol{x} \text{ is an } \textbf{arbitrary constant}\text{)}$$

$$= [R^2\theta - x^2 tan\theta]_0^{\cos^{-1}\left(\frac{x}{R}\right)} = R^2 \cos^{-1}\left(\frac{x}{R}\right) - x^2 tan\left(\cos^{-1}\left(\frac{x}{R}\right)\right)$$

$$= R^2 \cos^{-1}\left(\frac{x}{R}\right) - x^2\left(\frac{\sqrt{R^2 - x^2}}{x}\right) = R^2 \cos^{-1}\left(\frac{x}{R}\right) - x\sqrt{R^2 - x^2}$$

Hence, the **area of circular base** $((A_b)_{minor})$ **of the minor part removed from original right circular cone**,

$$\boldsymbol{(A_b)_{minor} = R^2 \cos^{-1}\left(\frac{x}{R}\right) - x\sqrt{R^2 - x^2} \qquad \forall\ 0 \le x \le R}$$

Hence, the **area of circular base** $\left((A_b)_{major}\right)$ **of cut right circular cone (i.e. major part left over from original cone),**

$$(A_b)_{major} = \text{total area of circular base of original right circular cone} - \text{ area of base of removed part}$$

$$= \pi R^2 - (A_b)_{minor}$$

$$\boldsymbol{(A_b)_{major} = \pi R^2 + x\sqrt{R^2 - x^2} - R^2 \cos^{-1}\left(\frac{x}{R}\right)}$$

Important deductions: 1. If the cutting plane passes through the symmetrical (longitudinal) axis of the original right circular cone then substituting $x = 0$ in the above expressions, we get

$$(A_b)_{minor} = \lim_{x\to 0} R^2\cos^{-1}\left(\frac{x}{R}\right) - x\sqrt{R^2 - x^2} = \frac{1}{2}\pi R^2 = \frac{1}{2}(\text{area of circular base of original cone})$$

$$(A_b)_{major} = \pi R^2 - (A_b)_{minor} = \pi R^2 - \frac{1}{2}\pi R^2 = \frac{1}{2}\pi R^2 = \frac{1}{2}(\text{area of circular base of original cone})$$

The above deduction is obviously true that a plane passing through the symmetrical (longitudinal) axis divides the right circular cone into two equal parts each having area of base half that of the original cone.

2. Substituting $x = R$ in the above expressions, we get

$$(A_b)_{minor} = \lim_{x\to R} R^2\cos^{-1}\left(\frac{x}{R}\right) - x\sqrt{R^2 - x^2} = 0 \ \ \& \ (A_b)_{major} = \pi R^2$$

It is obvious that the cutting plane, parallel to the symmetrical axis & tangent to the base circle of a right circular cone doesn't remove any part of cone.

Conclusion: When a right circular cone of vertical height H & base radius R is thoroughly cut by a plane parallel to its symmetrical (longitudinal) axis at a distance x $(0 \le x \le R)$, then we get two parts each with either a hyperbolic section or an isosceles triangular section. The volume & surface area of these two parts (usually unequal parts) of cone are computed as tabulated below

Parameter	**Minor part (removed from original right circular cone)**	**Major part/cut cone (left over)**
Volume	$V = \frac{H}{3R}\left[R^3\cos^{-1}\left(\frac{x}{R}\right) - 2Rx\sqrt{R^2 - x^2} + x^3\ln\left(\frac{R + \sqrt{R^2 - x^2}}{x}\right)\right]$	$\frac{1}{3}\pi R^2 H - V$
Area of curved surface	$S = \frac{(R^2 - x^2)\sqrt{H^2 + R^2}}{R}\cos^{-1}\left(\frac{x}{R}\right)$	$\pi R\sqrt{H^2 + R^2} - S$
Area of hyperbolic section	$H\sqrt{R^2 - x^2} + \frac{Hx^2}{R}\ln\left(\frac{x}{R + \sqrt{R^2 - x^2}}\right)$	$H\sqrt{R^2 - x^2} + \frac{Hx^2}{R}\ln\left(\frac{x}{R + \sqrt{R^2 - x^2}}\right)$
Area of circular base	$A_{base} = R^2\cos^{-1}\left(\frac{x}{R}\right) - x\sqrt{R^2 - x^2}$	$\pi R^2 - A_{base}$

*All the articles above have been derived by the author by using **simple geometry, trigonometry & calculus**. All above formula are the most generalised expressions which can be used for computing the volume & surface area of minor & major parts usually each with hyperbolic section obtained by cutting a right circular cone with a plane parallel to its symmetrical (longitudinal) axis.*

Note: Above articles had been derived & *illustrated by* ***Mr H.C. Rajpoot (B Tech, Mechanical Engineering)***

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