$\textit{Solid angle subtended by a tetrahedron at its vertex}$ & $\textit{solid angle subtended by a triangle at the origin}$

Harish Chandra Rajpoot *Aug, 2016*

M.M.M. University of Technology, Gorakhpur-273010 (UP), India

Here we are interested in finding out the solid angle subtended by any tetrahedron at its vertex given the angles between the consecutive lateral edges meeting at that vertex. In general, the solid angle subtended by any tetrahedron at its vertex is the function of angles between lateral edges meeting at that vertex. Consider a tetrahedron OPQR such that α, β & γ are the angle between the consecutive lateral edges OP & OQ, OQ & OR, and OP & OR respectively meeting at the vertex O (as shown in the figure 1).

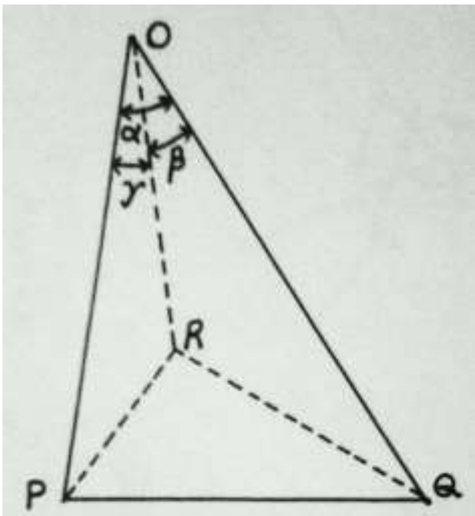

Figure 1: $\boldsymbol{\alpha}, \boldsymbol{\beta}$ & $\boldsymbol{\gamma}$ are the angles between lateral edges OP, OQ & OR meeting at vertex O of tetrahedron OPQR

Now, draw a spherical surface of radius R with centre at the vertex O of tetrahedron OPQR such that it intersects lateral edges OP, OQ, OR at the points A, B & C respectively (As shown in the figure 2 below). Join the points of intersection A, B & C by dotted straight lines & by great circle arcs to obtain a plane ΔABC & a spherical ΔABC. Now, we have (see figure 2 below)

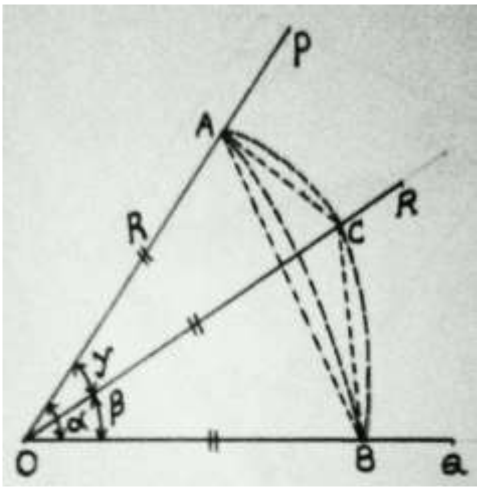

Figure 2: Plane $\boldsymbol{\Delta ABC}$ & spherical $\boldsymbol{\Delta ABC}$ obtained by the intersection of spherical surface & lateral edges of tetrahedron OPQR, subtend equal solid angle at centre O of sphere

$$\text{In isosceles } \Delta AOB, \qquad \Rightarrow AB = 2R\sin\frac{\alpha}{2}$$

$$\text{In isosceles } \Delta BOC, \qquad \Rightarrow BC = 2R\sin\frac{\beta}{2}$$

$$\text{In isosceles } \Delta AOC, \qquad \Rightarrow AC = 2R\sin\frac{\gamma}{2}$$

Now, applying cosine formula in plane ΔABC,

$$\cos\angle C = \frac{(BC)^2 + (CA)^2 - (AB)^2}{2(BC)(CA)} = \frac{\left(2R\sin\frac{\beta}{2}\right)^2 + \left(2R\sin\frac{\gamma}{2}\right)^2 - \left(2R\sin\frac{\alpha}{2}\right)^2}{2\left(2R\sin\frac{\beta}{2}\right)\left(2R\sin\frac{\gamma}{2}\right)}$$

$$\cos\angle C = \frac{\sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} - \sin^2\frac{\alpha}{2}}{2\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}$$

The plane angle θ_p between the chords of any two great circle arcs meeting each other at an angle θ on a spherical surface are co-related by **HCR's cosine formula** as follows

$$\boldsymbol{\cos\theta_p = \sin\frac{\delta}{2}\sin\frac{\varphi}{2} + \cos\frac{\delta}{2}\cos\frac{\varphi}{2}\cos\theta} \qquad \forall\ 0 < \delta, \varphi, \theta \le \pi \Rightarrow 0 \le \theta_p < \theta$$

$$\Rightarrow \boldsymbol{cos\theta} = \frac{\boldsymbol{cos\theta_p - sin\frac{\alpha}{2}sin\frac{\beta}{2}}}{\boldsymbol{cos\frac{\alpha}{2}cos\frac{\beta}{2}}}$$

Where, $\boldsymbol{\delta}$ & $\boldsymbol{\varphi}$ are the angles subtended by two great circle arcs at the centre of spherical surface.

Now,

$\theta_p = \text{plane angle between chords BC \& AC} = \angle \text{C}$,

$\delta = \text{angle subtended by great arc BC at the centre O} = \beta$,

$\varphi = \text{angle subtended by great arc AC at the centre O} = \gamma$

Setting the corresponding values in the above cosine formula, we get the angle θ_1 between the great circle arcs BC & AC as follows

$$cos\theta_1 = \frac{cos\angle C - sin\frac{\beta}{2}sin\frac{\gamma}{2}}{cos\frac{\beta}{2}cos\frac{\gamma}{2}}$$

$$= \frac{\frac{sin^2\frac{\beta}{2} + sin^2\frac{\gamma}{2} - sin^2\frac{\alpha}{2}}{2sin\frac{\beta}{2}sin\frac{\gamma}{2}} - sin\frac{\beta}{2}sin\frac{\gamma}{2}}{cos\frac{\beta}{2}cos\frac{\gamma}{2}}$$

$$= \frac{sin^2\frac{\beta}{2} + sin^2\frac{\gamma}{2} - sin^2\frac{\alpha}{2} - 2sin^2\frac{\beta}{2}sin^2\frac{\gamma}{2}}{2sin\frac{\beta}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2}cos\frac{\gamma}{2}}$$

$$= \frac{\left(sin^2\frac{\beta}{2} - sin^2\frac{\beta}{2}sin^2\frac{\gamma}{2}\right) + \left(sin^2\frac{\gamma}{2} - sin^2\frac{\beta}{2}sin^2\frac{\gamma}{2}\right) - sin^2\frac{\alpha}{2}}{\frac{1}{2}\left(2sin\frac{\beta}{2}cos\frac{\beta}{2}\right)\left(2sin\frac{\gamma}{2}cos\frac{\gamma}{2}\right)}$$

$$= \frac{sin^2\frac{\beta}{2}\left(1 - sin^2\frac{\gamma}{2}\right) + sin^2\frac{\gamma}{2}\left(1 - sin^2\frac{\beta}{2}\right) - sin^2\frac{\alpha}{2}}{\frac{1}{2}(sin\beta)(sin\gamma)}$$

$$= \frac{2\left(sin^2\frac{\beta}{2}cos^2\frac{\gamma}{2} + cos^2\frac{\beta}{2}sin^2\frac{\gamma}{2} - sin^2\frac{\alpha}{2}\right)}{sin\beta sin\gamma}$$

$$= \frac{2\left(sin^2\frac{\beta}{2}cos^2\frac{\gamma}{2} + cos^2\frac{\beta}{2}sin^2\frac{\gamma}{2} + 2sin\frac{\beta}{2}cos\frac{\gamma}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2} - 2sin\frac{\beta}{2}cos\frac{\gamma}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2} - sin^2\frac{\alpha}{2}\right)}{sin\beta sin\gamma}$$

$$= \frac{2\left(\left(sin\frac{\beta}{2}cos\frac{\gamma}{2} + cos\frac{\beta}{2}sin\frac{\gamma}{2}\right)^2 - 2sin\frac{\beta}{2}cos\frac{\gamma}{2}cos\frac{\beta}{2}sin\frac{\gamma}{2} - sin^2\frac{\alpha}{2}\right)}{sin\beta sin\gamma}$$

$$= \frac{2\left(\left(sin\left(\frac{\beta+\gamma}{2}\right)\right)^2 - \frac{1}{2}\left(2sin\frac{\beta}{2}cos\frac{\beta}{2}\right)\left(2sin\frac{\gamma}{2}cos\frac{\gamma}{2}\right) - sin^2\frac{\alpha}{2}\right)}{sin\beta sin\gamma}$$

$$= \frac{2\left(sin^2\left(\frac{\beta+\gamma}{2}\right) - \frac{1}{2}sin\beta sin\gamma - sin^2\frac{\alpha}{2}\right)}{sin\beta sin\gamma}$$

$$= \frac{2sin^2\left(\frac{\beta+\gamma}{2}\right) - sin\beta sin\gamma - 2sin^2\frac{\alpha}{2}}{sin\beta sin\gamma}$$

$$= \frac{\big(1 - cos(\beta+\gamma)\big) - sin\beta sin\gamma - (1 - cos\alpha)}{sin\beta sin\gamma}$$

$$= \frac{cos\alpha - cos(\beta+\gamma) - sin\beta sin\gamma}{sin\beta sin\gamma}$$

$$= \frac{cos\alpha - (cos\beta cos\gamma - sin\beta sin\gamma) - sin\beta sin\gamma}{sin\beta sin\gamma}$$

$$= \frac{cos\alpha - cos\beta cos\gamma + sin\beta sin\gamma - sin\beta sin\gamma}{sin\beta sin\gamma}$$

$$cos\theta_1 = \frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}$$

$$\theta_1 = \cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right)$$

Similarly, the angle θ_2 between the great circle arcs AC & AB can be found out as follows

$$\theta_2 = \cos^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right)$$

Similarly, the angle θ_3 between the great circle arcs AB & BC can be found out as follows

$$\theta_3 = \cos^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right)$$

Thus, θ_1, θ_2 & θ_3 are the angles between great circle arcs AC & BC, AC & AB, and AB & BC respectively (see figure 2 above) which are also the interior angles of spherical triangle ABC.

Now, the solid angle subtended by the tetrahedron OPQR at its apex O

$$\omega = \text{solid angle subtended by spherical triangle ABC at the centre O of spherical surface of radius R}$$

$$= \frac{\text{Area covered by spherical triangle ABC having interior angles } \theta_1, \theta_2 \text{ \& } \theta_3}{(\text{radius of spherical surface})^2}$$

$$= \frac{(\theta_1 + \theta_2 + \theta_3 - \pi)R^2}{R^2} = \theta_1 + \theta_2 + \theta_3 - \pi$$

$$= \cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) + \cos^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right) + \cos^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right) - \pi$$

$$= \cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) - \left(\frac{\pi}{2} - \cos^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right)\right) - \left(\frac{\pi}{2} - \cos^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right)\right)$$

$$= \cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) - \sin^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right) - \sin^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right)$$

Hence, the **solid angle subtended by any tetrahedron at its vertex** is given by the general formula

$$\boldsymbol{\omega = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) - \sin^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}\right) - \sin^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right)}$$

Where, $\alpha, \beta\ \&\ \gamma$ are the angles between consecutive lateral edges meeting at the concerned vertex of tetrahedron $(\forall\ \ \boldsymbol{0 < \alpha, \beta, \gamma < \pi\ \&\ \alpha + \beta + \gamma < 2\pi})$

It is worth noticing that the above formula is valid in all the cases. It gives the maximum value of solid angle as 2π sr when the vertex of tetrahedron lies on the plane of base which is true because the solid angle subtended by any plane at each point on it within its boundary is always 2π sr.

Solid angle subtended by any triangle at the origin in 3D co-ordinate system given the position vectors of the vertices: Let's consider any triangle ABC in 3D space such that $\vec{a}, \vec{b}\ \&\ \vec{c}$ are the position vectors of the vertices A, B & C respectively (as shown in the figure 3).

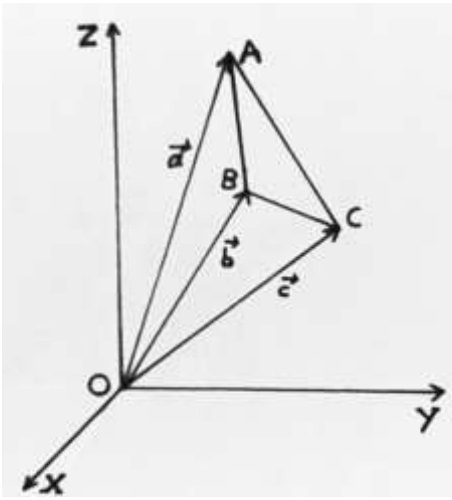

Figure 3: $\boldsymbol{\vec{a}, \vec{b}\ \&\ \vec{c}}$ **are the position vectors of vertices A, B & C of** $\boldsymbol{\Delta ABC}$ **in 3D space**

Now, let $\angle AOB = \alpha$, $\angle BOC = \beta\ \&\ \angle AOC = \gamma$ then we use vector product & scalar product of two vectors, the vector product $\vec{a}\ \&\ \vec{b}$ is given as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\alpha\hat{n} \Rightarrow \sin\alpha = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

The scalar product $\vec{a}\ \&\ \vec{b}$ is given as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\alpha \Rightarrow \cos\alpha = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Similarly, it can be found out that

$$\sin\beta = \frac{|\vec{b} \times \vec{c}|}{|\vec{b}||\vec{c}|}, \quad \cos\beta = \frac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|}, \quad \sin\gamma = \frac{|\vec{c} \times \vec{a}|}{|\vec{c}||\vec{a}|}, \quad \cos\gamma = \frac{\vec{c} \cdot \vec{a}}{|\vec{c}||\vec{a}|}$$

Now, the solid angle subtended by ΔABC at the origin O is equal to the solid angle subtended by the tetrahedron OABC at the apex O such that $\alpha, \beta\ \&\ \gamma$ are the angles between consecutive lateral edges AO, BO & CO at the vertex O. The solid angle subtended by the tetrahedron is given by the general formula

$$\omega = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) - \sin^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}\right) - \sin^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right)$$

Substituting the values of $\sin\alpha, \cos\alpha, \sin\beta, \cos\beta, \sin\gamma, \cos\gamma$ in the above formula, one should get

$$\omega = \cos^{-1}\left(\frac{\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} - \frac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|} \cdot \frac{\vec{c} \cdot \vec{a}}{|\vec{c}||\vec{a}|}}{\frac{|\vec{b} \times \vec{c}|}{|\vec{b}||\vec{c}|} \frac{|\vec{c} \times \vec{a}|}{|\vec{c}||\vec{a}|}}\right) - \sin^{-1}\left(\frac{\frac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|} - \frac{\vec{c} \cdot \vec{a}}{|\vec{c}||\vec{a}|} \cdot \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}}{\frac{|\vec{c} \times \vec{a}|}{|\vec{c}||\vec{a}|} \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}}\right) - \sin^{-1}\left(\frac{\frac{\vec{c} \cdot \vec{a}}{|\vec{c}||\vec{a}|} - \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \cdot \frac{\vec{b} \cdot \vec{c}}{|\vec{b}||\vec{c}|}}{\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} \frac{|\vec{b} \times \vec{c}|}{|\vec{b}||\vec{c}|}}\right)$$

$$\omega = \cos^{-1}\left(\frac{(\vec{a} \cdot \vec{b})|\vec{c}|^2 - (\vec{b} \cdot \vec{c})(\vec{c} \cdot \vec{a})}{|\vec{b} \times \vec{c}||\vec{c} \times \vec{a}|}\right) - \sin^{-1}\left(\frac{(\vec{b} \cdot \vec{c})|\vec{a}|^2 - (\vec{c} \cdot \vec{a})(\vec{a} \cdot \vec{b})}{|\vec{c} \times \vec{a}||\vec{a} \times \vec{b}|}\right) - \sin^{-1}\left(\frac{(\vec{c} \cdot \vec{a})|\vec{b}|^2 - (\vec{a} \cdot \vec{b})(\vec{b} \cdot \vec{c})}{|\vec{a} \times \vec{b}||\vec{b} \times \vec{c}|}\right)$$

Above is the general formula to compute the solid angle subtended by any triangle at the origin given the position vectors of its vertices as $\vec{a}$, $\vec{b}$ & $\vec{c}$.

Illustrative Examples

Example 1: **Compute the solid angle subtended by a tetrahedron at its vertex such that the angles between the consecutive edges meeting at the same vertex are $\mathbf{30^o, 40^o}$ & $\mathbf{50^o}$.**

Sol. In this case, $\alpha+\beta+\gamma = 30^o + 40^o + 50^o = 120^o < 360^o$ (true tetrahedron)

The solid angle ω subtended by a tetrahedron at its vertex is given by general formula

$$\omega = \cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) - \sin^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right) - \sin^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right)$$

Substituting the corresponding values, $\alpha = 30^o$, $\beta = 40^o$ & $\gamma = 50^o$ in the above formula, one should get

$$\boldsymbol{\omega} = \cos^{-1}\left(\frac{cos30^o - cos40^o cos50^o}{sin40^o sin50^o}\right) - \sin^{-1}\left(\frac{cos40^o - cos50^o cos30^o}{sin50^o sin30^o}\right) - \sin^{-1}\left(\frac{cos50^o - cos30^o cos40^o}{sin30^o sin40^o}\right)$$

$$\omega = 0.709372913 - 0.578342569 - (-0.06422191)$$

$$= 0.773594823 - 0.578342569 = \mathbf{0.195252254\ sr}$$

Example 2: **Compute the solid angle subtended by a tetrahedron at its vertex such that the angles between the consecutive edges meeting at the same vertex are $\mathbf{90^o, 90^o}$ & $\mathbf{90^o}$.**

Sol. In this case, $\alpha+\beta+\gamma = 90^o + 90^o + 90^o = 270^o < 360^o$ (true tetrahedron)

The solid angle ω subtended by a tetrahedron at its vertex is given by general formula

$$\omega = \cos^{-1}\left(\frac{cos\alpha - cos\beta cos\gamma}{sin\beta sin\gamma}\right) - \sin^{-1}\left(\frac{cos\beta - cos\gamma cos\alpha}{sin\gamma sin\alpha}\right) - \sin^{-1}\left(\frac{cos\gamma - cos\alpha cos\beta}{sin\alpha sin\beta}\right)$$

Substituting the corresponding values, $\alpha = 90^o$, $\beta = 90^o$ & $\gamma = 90^o$ in the above formula, we get

$$\boldsymbol{\omega} = \cos^{-1}\left(\frac{cos90^o - cos90^o cos90^o}{sin90^o sin90^o}\right) - \sin^{-1}\left(\frac{cos90^o - cos90^o cos90^o}{sin90^o sin90^o}\right) - \sin^{-1}\left(\frac{cos90^o - cos90^o cos90^o}{sin90^o sin90^o}\right)$$

$$\omega = \cos^{-1}(0) - \sin^{-1}(0) - \sin^{-1}(0) = \frac{\pi}{2} - 0 - 0 = \boldsymbol{\frac{\pi}{2}}\ \mathbf{sr}$$

It is obvious that the solid angle subtended by an octant at the origin is $\frac{4\pi}{8} = \frac{\pi}{2}$ sr which is equal to the above value of solid angle subtended by a tetrahedron which is equivalent to an octant. Thus the result is verified.

Example 3: **Compute the solid angle subtended by a tetrahedron at its vertex such that the angles between the consecutive edges meeting at the same vertex are $\mathbf{90^o, 120^o}$ & $\mathbf{150^o}$.**

Sol. In this case, $\alpha+\beta+\gamma = 90^o + 120^o + 150^o = 360^o \nless 360^o$ (not a tetrahedron)

The solid angle ω subtended by a tetrahedron at its vertex is given by general formula

$$\omega=\cos^{-1}\left(\frac{cos\alpha-cos\beta cos\gamma}{sin\beta sin\gamma}\right)-\sin^{-1}\left(\frac{cos\beta-cos\gamma cos\alpha}{sin\gamma sin\alpha}\right)-\sin^{-1}\left(\frac{cos\gamma-cos\alpha cos\beta}{sin\alpha sin\beta}\right)$$

Substituting the corresponding values, $\alpha=90^o$, $\beta=120^o$ & $\gamma=150^o$ in the above formula, we get solid angle subtended by a plane at any point on it within its boundary

$$\boldsymbol{\omega}=\cos^{-1}\left(\frac{cos90^o-cos120^o cos150^o}{sin120^o sin150^o}\right)-\sin^{-1}\left(\frac{cos120^o-cos150^o cos90^o}{sin150^o sin90^o}\right)-\sin^{-1}\left(\frac{cos150^o-cos90^o cos120^o}{sin90^o sin120^o}\right)$$

$$\omega=\cos^{-1}(-1)-\sin^{-1}(-1)-\sin^{-1}(-1)$$

$$=\pi-\left(-\frac{\pi}{2}\right)-\left(-\frac{\pi}{2}\right)=\pi+\frac{\pi}{2}+\frac{\pi}{2}=\boldsymbol{2\pi}\ \mathbf{sr}$$

Above result is true for any plane subtending a solid angle of 2π sr at each point on it within its boundary.

Example 4: **Compute the solid angle subtended at the origin by a triangle having its vertices at the points $(\mathbf{1},\mathbf{2},\mathbf{3}),(-\mathbf{2},-\mathbf{3},\mathbf{1})$ & $(-\mathbf{3},\mathbf{4},\mathbf{5})$ in 3D coordinate system.**

Sol. In this case, the position vectors of vertices: $\vec{a}=\hat{\imath}+2\hat{\jmath}+3\hat{k}$, $\vec{b}=-2\hat{\imath}-3\hat{\jmath}+\hat{k}$ & $\vec{c}=-3\hat{\imath}+4\hat{\jmath}+5\hat{k}$

$$|\vec{a}|=\left|\hat{\imath}+2\hat{\jmath}+3\hat{k}\right|=\sqrt{14},\quad \left|\vec{b}\right|=\left|-2\hat{\imath}-3\hat{\jmath}+\hat{k}\right|=\sqrt{14},\quad |\vec{c}|=\left|-3\hat{\imath}+4\hat{\jmath}+5\hat{k}\right|=5\sqrt{2}$$

$$\vec{a}\cdot\vec{b}=\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\cdot\left(-2\hat{\imath}-3\hat{\jmath}+\hat{k}\right)=-5$$

$$\vec{b}\cdot\vec{c}=\left(-2\hat{\imath}-3\hat{\jmath}+\hat{k}\right)\cdot\left(-3\hat{\imath}+4\hat{\jmath}+5\hat{k}\right)=-1$$

$$\vec{c}\cdot\vec{a}=\left(-3\hat{\imath}+4\hat{\jmath}+5\hat{k}\right)\cdot\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)=20$$

$$\left|\vec{a}\times\vec{b}\right|=\left|\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\times\left(-2\hat{\imath}-3\hat{\jmath}+\hat{k}\right)\right|=\left|11\hat{\imath}-7\hat{\jmath}+\hat{k}\right|=3\sqrt{19}$$

$$\left|\vec{b}\times\vec{c}\right|=\left|\left(-2\hat{\imath}-3\hat{\jmath}+\hat{k}\right)\times\left(-3\hat{\imath}+4\hat{\jmath}+5\hat{k}\right)\right|=\left|-19\hat{\imath}+7\hat{\jmath}-17\hat{k}\right|=\sqrt{699}$$

$$|\vec{c}\times\vec{a}|=\left|\left(-3\hat{\imath}+4\hat{\jmath}+5\hat{k}\right)\times\left(\hat{\imath}+2\hat{\jmath}+3\hat{k}\right)\right|=\left|2\hat{\imath}+14\hat{\jmath}-10\hat{k}\right|=10\sqrt{3}$$

The solid angle ω subtended by a triangle at the origin given the position vectors of its vertices is given as

$$\omega=\cos^{-1}\left(\frac{(\vec{a}\cdot\vec{b})|\vec{c}|^2-(\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})}{|\vec{b}\times\vec{c}||\vec{c}\times\vec{a}|}\right)-\sin^{-1}\left(\frac{(\vec{b}\cdot\vec{c})|\vec{a}|^2-(\vec{c}\cdot\vec{a})(\vec{a}\cdot\vec{b})}{|\vec{c}\times\vec{a}||\vec{a}\times\vec{b}|}\right)-\sin^{-1}\left(\frac{(\vec{c}\cdot\vec{a})\left|\vec{b}\right|^2-(\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})}{|\vec{a}\times\vec{b}||\vec{b}\times\vec{c}|}\right)$$

Now, setting all the corresponding values in above formula, the solid angle subtended by triangle at the origin

$$\boldsymbol{\omega}=\cos^{-1}\left(\frac{(-5)\left(5\sqrt{2}\right)^2-(-1)(20)}{\left(\sqrt{699}\right)\left(10\sqrt{3}\right)}\right)-\sin^{-1}\left(\frac{(-1)\left(\sqrt{14}\right)^2-(20)(-5)}{\left(10\sqrt{3}\right)\left(3\sqrt{19}\right)}\right)-\sin^{-1}\left(\frac{(20)\left(\sqrt{14}\right)^2-(-5)(-1)}{\left(3\sqrt{19}\right)\left(\sqrt{699}\right)}\right)$$

$$=\cos^{-1}\left(\frac{-23}{3\sqrt{233}}\right)-\sin^{-1}\left(\frac{43}{15\sqrt{57}}\right)-\sin^{-1}\left(\frac{275}{3\sqrt{13281}}\right)$$

$$=2.097006736-0.389471206-0.919698889=\mathbf{0.787836641\ sr}$$

Example 5: **Prove that the solid angle subtended at the origin by any triangle having its vertices at the points $(\boldsymbol{x}, \mathbf{0}, \mathbf{0}), (\mathbf{0}, \boldsymbol{y}, \mathbf{0})$ & $(\mathbf{0}, \mathbf{0}, \boldsymbol{z})$ on the coordinate axes in 3D space is always $\boldsymbol{\pi}/_{\mathbf{2}}$ sr.** $(\forall\ \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbf{R})$

Sol. Let the triangle ABC have its vertices at the points $A(x, 0, 0), B(0, y, 0)$ & $C(0, 0, z)$ lying on the coordinate axes x, y & z axes respectively. (as shown in the figure 4)

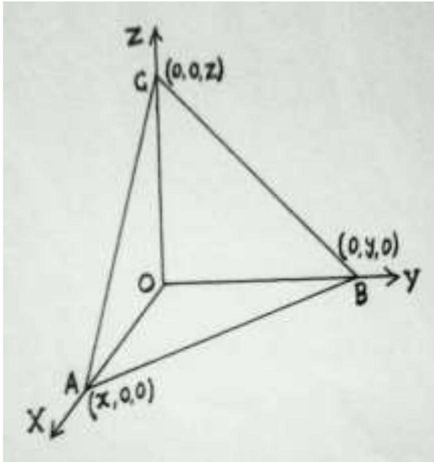

Figure 4: $\boldsymbol{A}(\boldsymbol{x}, \mathbf{0}, \mathbf{0}), \boldsymbol{B}(\mathbf{0}, \boldsymbol{y}, \mathbf{0})$ & $\boldsymbol{C}(\mathbf{0}, \mathbf{0}, \boldsymbol{z})$ **are the vertices of $\Delta\boldsymbol{ABC}$ lying on the coordinate axes x, y, & z respectively.**

In this case, the position vectors of vertices of ΔABC: $\vec{a} = x\hat{\imath}$, $\vec{b} = y\hat{\jmath}$ & $\vec{c} = z\hat{k}$

The solid angle ω subtended by a triangle at the origin given the position vectors of its vertices is given as

$$\omega = \cos^{-1}\left(\frac{(\vec{a}\cdot\vec{b})|\vec{c}|^2 - (\vec{b}\cdot\vec{c})(\vec{c}\cdot\vec{a})}{|\vec{b}\times\vec{c}||\vec{c}\times\vec{a}|}\right) - \sin^{-1}\left(\frac{(\vec{b}\cdot\vec{c})|\vec{a}|^2 - (\vec{c}\cdot\vec{a})(\vec{a}\cdot\vec{b})}{|\vec{c}\times\vec{a}||\vec{a}\times\vec{b}|}\right) - \sin^{-1}\left(\frac{(\vec{c}\cdot\vec{a})|\vec{b}|^2 - (\vec{a}\cdot\vec{b})(\vec{b}\cdot\vec{c})}{|\vec{a}\times\vec{b}||\vec{b}\times\vec{c}|}\right)$$

Now, setting the values of all the vectors in above formula, the solid angle subtended by triangle at the origin

$$\omega = \cos^{-1}\left(\frac{(x\hat{\imath}\cdot y\hat{\jmath})|z\hat{k}|^2 - (y\hat{\jmath}\cdot z\hat{k})(z\hat{k}\cdot x\hat{\imath})}{|y\hat{\jmath}\times z\hat{k}||z\hat{k}\times x\hat{\imath}|}\right) - \sin^{-1}\left(\frac{(y\hat{\jmath}\cdot z\hat{k})|x\hat{\imath}|^2 - (z\hat{k}\cdot x\hat{\imath})(x\hat{\imath}\cdot y\hat{\jmath})}{|z\hat{k}\times x\hat{\imath}||x\hat{\imath}\times y\hat{\jmath}|}\right) - \sin^{-1}\left(\frac{(z\hat{k}\cdot x\hat{\imath})|y\hat{\jmath}|^2 - (x\hat{\imath}\cdot y\hat{\jmath})(y\hat{\jmath}\cdot z\hat{k})}{|x\hat{\imath}\times y\hat{\jmath}||y\hat{\jmath}\times z\hat{k}|}\right)$$

$$= \cos^{-1}\left(\frac{xyz^2(\hat{\imath}\cdot\hat{\jmath}) - xyz^2(\hat{\jmath}\cdot\hat{k})(\hat{k}\cdot\hat{\imath})}{xyz^2|\hat{\jmath}\times\hat{k}||\hat{k}\times\hat{\imath}|}\right) - \sin^{-1}\left(\frac{x^2yz(\hat{\jmath}\cdot\hat{k}) - x^2yz(\hat{k}\cdot\hat{\imath})(\hat{\imath}\cdot\hat{\jmath})}{x^2yz|\hat{k}\times\hat{\imath}||\hat{\imath}\times\hat{\jmath}|}\right) - \sin^{-1}\left(\frac{xy^2z(\hat{k}\cdot\hat{\imath}) - xy^2z(\hat{\imath}\cdot\hat{\jmath})(\hat{\jmath}\cdot\hat{k})}{xy^2z|\hat{\imath}\times\hat{\jmath}||\hat{\jmath}\times\hat{k}|}\right)$$

$$= \cos^{-1}\left(\frac{(\hat{\imath}\cdot\hat{\jmath}) - (\hat{\jmath}\cdot\hat{k})(\hat{k}\cdot\hat{\imath})}{|\hat{\jmath}\times\hat{k}||\hat{k}\times\hat{\imath}|}\right) - \sin^{-1}\left(\frac{(\hat{\jmath}\cdot\hat{k}) - (\hat{k}\cdot\hat{\imath})(\hat{\imath}\cdot\hat{\jmath})}{|\hat{k}\times\hat{\imath}||\hat{\imath}\times\hat{\jmath}|}\right) - \sin^{-1}\left(\frac{(\hat{k}\cdot\hat{\imath}) - (\hat{\imath}\cdot\hat{\jmath})(\hat{\jmath}\cdot\hat{k})}{|\hat{\imath}\times\hat{\jmath}||\hat{\jmath}\times\hat{k}|}\right)$$

$$= \cos^{-1}(0) - \sin^{-1}(0) - \sin^{-1}(0) = \frac{\pi}{2} - 0 - 0 = \boldsymbol{\frac{\pi}{2}}\ \mathbf{sr}$$

It is obvious that the **solid angle subtended at the origin by any triangle having its vertices on three coordinate axes is always $\boldsymbol{\pi}/_{\mathbf{2}}$ sr** . It also shows that the solid angle subtended by an octant at the origin is $\frac{4\pi}{8} = \frac{\pi}{2}$ sr which is equal to the above value of solid angle subtended at the origin by a triangle with vertices on the coordinate axes which is equivalent to an octant. Thus the result is proved & verified.

Conclusion: It can be concluded that above general formula gives the correct values of the solid angle subtended by any tetrahedron at its vertex when the angles between consecutive lateral edges meeting at that vertex are known because there is no approximation in the formula. This is an analytic & precision formula to compute the correct value of solid angle subtended by a triangle at the origin which is equally applicable in all the cases given the position vectors of all three vertices in 3D coordinate system.

Note: Above articles had been derived *& illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)*

M.M.M. University of Technology, Gorakhpur-273010 (UP) India *Aug, 2016*

Email: rajpootharishchandra@gmail.com
Author's Home Page: https://notionpress.com/author/HarishChandraRajpoot