

Mathematical Analysis of Disphenoid (Isosceles Tetrahedron)

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1. Introduction: We very well know that **disphenoid** is a **tetrahedron** having **four congruent faces** each as an **acute-angled triangle**. Each pair of **opposite edges** of a **disphenoid** is **equal in length** hence it is also called **isosceles tetrahedron** (as shown in the figure1). Solid angle subtended by a disphenoid at each of its vertices is equal i.e. solid angle subtended by each face at its opposite vertex is equal because each face has the same area & is at an equal normal distance from its opposite vertex. Here we are interested to derive the general formula to compute volume, surface area, radii of inscribed & circumscribed spheres, coordinates of four vertices & the coordinates of in-centre, circum-centre & centroid of a disphenoid using 3D coordinate geometry.

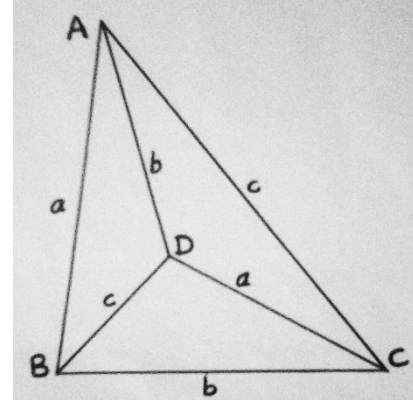


Figure 1: A disphenoid ABCD has each pair of opposite edges equal in length ($AB = CD = a, AD = BC = b, AC = BD = c$)

2. Analysis of disphenoid (isosceles tetrahedron): Consider a disphenoid ABCD having four congruent triangular faces each of sides a, b & c in 3D space optimally such that its triangular face ABC lies on XY-plane so that vertex A is at origin & vertex B is on the x-axis. (As shown in the figure 2)

Applying cosine formula in ΔABC as follows

$$\cos A = \frac{(AB)^2 + (AC)^2 - (BC)^2}{2(AB)(AC)} = \frac{a^2 + c^2 - b^2}{2ac}$$

In right ΔANC , $AN = (AC)\cos A = c \cos A = c \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$

$$AN = \frac{a^2 + c^2 - b^2}{2a} = \mathbf{x \text{ coordinate of vertex C}}$$

Now, drop a perpendicular CN from vertex C to the side AB, the area of ΔABC is as

$$\Delta = \frac{1}{2}(\text{base})(\text{normal height}) = \frac{1}{2}(a)(CN)$$

$$CN = \frac{2\Delta}{a} = \mathbf{y \text{ coordinate of vertex C}}$$

Now, coordinates of vertex C are $(AN, CN, 0) \equiv \left(\frac{a^2 + c^2 - b^2}{2a}, \frac{2\Delta}{a}, 0 \right)$. Let the coordinates of vertex D be (x, y, z)

Now, using distance formula, the distance between the vertices $D(x, y, z)$ & $A(0, 0, 0)$ is given as

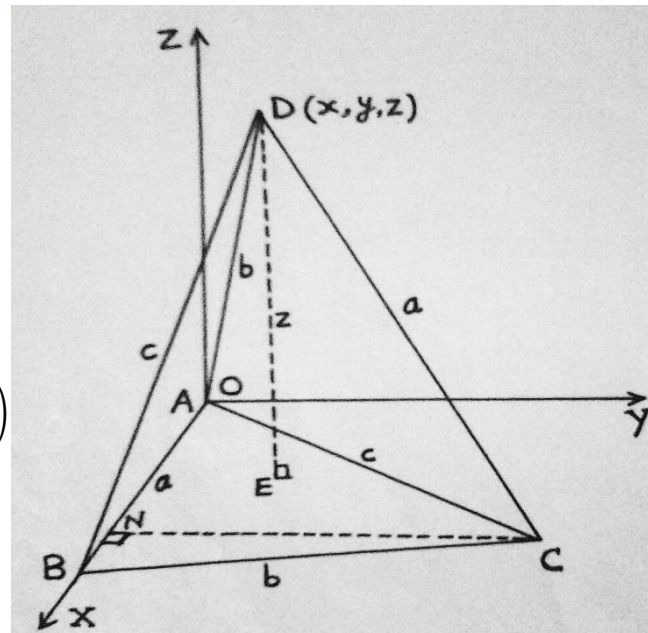


Figure 2: A disphenoid ABCD optimally has its vertex A at origin, vertex B on the x-axis, vertex C on the XY-plane & vertex D at (x, y, z) in the first octant in 3D space

$$DA = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = b$$

$$x^2 + y^2 + z^2 = b^2 \quad \dots \dots \dots (1)$$

Similarly, the distance between the vertices $D(x, y, z)$ & $B(a, 0, 0)$ is given as

$$DB = \sqrt{(x-a)^2 + (y-0)^2 + (z-0)^2} = c$$

$$(x-a)^2 + (y-0)^2 + (z-0)^2 = c^2$$

$$x^2 + y^2 + z^2 - 2ax + a^2 = c^2$$

Setting the value of $x^2 + y^2 + z^2$ from (1),

$$b^2 - 2ax + a^2 = c^2$$

$$x = \frac{a^2 + b^2 - c^2}{2a} \quad \dots \dots \dots (2)$$

Similarly, the distance between the vertices $D(x, y, z)$ & $C\left(\frac{a^2+c^2-b^2}{2a}, \frac{2\Delta}{a}, 0\right)$ is given as

$$DC = \sqrt{\left(x - \frac{a^2 + c^2 - b^2}{2a}\right)^2 + \left(y - \frac{2\Delta}{a}\right)^2 + (z-0)^2} = a$$

$$\left(x - \frac{a^2 + c^2 - b^2}{2a}\right)^2 + \left(y - \frac{2\Delta}{a}\right)^2 + (z-0)^2 = a^2$$

$$x^2 + y^2 + z^2 - 2\left(\frac{a^2 + c^2 - b^2}{2a}\right)x + \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 - 2\left(\frac{2\Delta}{a}\right)y + \left(\frac{2\Delta}{a}\right)^2 = a^2$$

Setting the value of $x^2 + y^2 + z^2$ from (1) & the value of x from (2),

$$b^2 - 2\left(\frac{a^2 + c^2 - b^2}{2a}\right)\left(\frac{a^2 + b^2 - c^2}{2a}\right) + \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 - \left(\frac{4\Delta}{a}\right)y + \left(\frac{2\Delta}{a}\right)^2 = a^2$$

$$\left(\frac{4\Delta}{a}\right)y = b^2 - a^2 - \frac{(a^2 + c^2 - b^2)(a^2 + b^2 - c^2)}{2a^2} + \frac{(a^2 + c^2 - b^2)^2}{4a^2} + \frac{4\Delta^2}{a^2}$$

$$\left(\frac{4\Delta}{a}\right)y = \frac{4a^2b^2 - 4a^4 - 2(a^2 + c^2 - b^2)(a^2 + b^2 - c^2) + (a^2 + c^2 - b^2)^2 + 16\Delta^2}{4a^2}$$

But from Heron's formula, $\Delta = \sqrt{s(s-a)(s-b)(s-c)} \Rightarrow 16\Delta^2 = 4a^2b^2 - (a^2 + b^2 - c^2)^2$ Setting the value of $16\Delta^2$ in above equation, we should get

$$\begin{aligned} \left(\frac{4\Delta}{a}\right)y &= \frac{4a^2b^2 - 4a^4 - 2(a^2 + c^2 - b^2)(a^2 + b^2 - c^2) + (a^2 + c^2 - b^2)^2 + 4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2} \\ &= \frac{8a^2b^2 - 4a^4 + (a^2 + c^2 - b^2)^2 - (a^2 + c^2 - b^2)(a^2 + b^2 - c^2) - (a^2 + c^2 - b^2)(a^2 + b^2 - c^2) - (a^2 + b^2 - c^2)^2}{4a^2} \\ &= \frac{8a^2b^2 - 4a^4 + (a^2 + c^2 - b^2)(a^2 + c^2 - b^2 - a^2 - b^2 + c^2) - (a^2 + b^2 - c^2)(a^2 + c^2 - b^2 + a^2 + b^2 - c^2)}{4a^2} \end{aligned}$$

$$\begin{aligned}
 &= \frac{8a^2b^2 - 4a^4 + (a^2 + c^2 - b^2)(2c^2 - 2b^2) - (a^2 + b^2 - c^2)(2a^2)}{4a^2} \\
 &= \frac{4a^2b^2 - 2a^4 + (a^2 + c^2 - b^2)(c^2 - b^2) - a^2(a^2 + b^2 - c^2)}{2a^2} \\
 \left(\frac{4\Delta}{a}\right)y &= \frac{4a^2b^2 - 2a^4 + a^2c^2 + c^4 - b^2c^2 - a^2b^2 - b^2c^2 + b^4 - a^4 - a^2b^2 + a^2c^2}{2a^2} \\
 y &= \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta} \dots \dots \dots (3)
 \end{aligned}$$

Now, setting the values of x & y from (2) & (3) respectively into (1), we should get

$$\begin{aligned}
 &\left(\frac{a^2 + b^2 - c^2}{2a}\right)^2 + \left(\frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}\right)^2 + z^2 = b^2 \\
 z &= \sqrt{b^2 - \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2 - \left(\frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}\right)^2} \\
 &= \sqrt{\frac{4a^2b^2 - (a^2 + b^2 - c^2)^2}{4a^2} - \frac{(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}{64a^2\Delta^2}} \\
 &= \sqrt{\frac{2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4}{4a^2} - \frac{(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}{64a^2\Delta^2}} \\
 &= \sqrt{\frac{16\Delta^2(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4) - (2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}{64a^2\Delta^2}} \\
 &= \frac{\sqrt{16\Delta^2(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4) - (2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}}{8a\Delta}
 \end{aligned}$$

But from Heron's formula, $\Delta = \sqrt{s(s-a)(s-b)(s-c)} \Rightarrow 16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$
 Setting the value of $16\Delta^2$ in above equation, we should get

$$\begin{aligned}
 z &= \frac{\sqrt{(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4)^2 - (2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}}{8a\Delta} \\
 &= \frac{\sqrt{(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 + 2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 - 2a^2b^2 + 2b^2c^2 - 2a^2c^2 + 3a^4 - b^4 - c^4)}}{8a\Delta} \\
 &= \frac{\sqrt{(4a^2b^2 + 4a^2c^2 - 4a^4)(4b^2c^2 + 2a^4 - 2b^4 - 2c^4)}}{8a\Delta} \\
 &= \frac{\sqrt{8a^2(b^2 + c^2 - a^2)(2b^2c^2 + a^4 - b^4 - c^4)}}{8a\Delta} \\
 &= \frac{2a\sqrt{2}\sqrt{(b^2 + c^2 - a^2)(a^4 - (b^2 - c^2)^2)}}{8a\Delta} \\
 &= \frac{\sqrt{(b^2 + c^2 - a^2)(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)}}{2\Delta\sqrt{2}}
 \end{aligned}$$

$$z = \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}} \dots \dots \dots (4)$$

Thus, setting the values of x, y & z from (2), (3) & (4), the coordinates of vertex D are given as

$$D \equiv \left(\frac{a^2 + b^2 - c^2}{2a}, \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}, \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}} \right)$$

The coordinates of four vertices of disphenoid (isosceles tetrahedron): The coordinates of all four vertices A, B, C & D of disphenoid ABCD with three unequal edges a, b & c for the optimal configuration in 3D space are given as $A(0, 0, 0), B(a, 0, 0), C\left(\frac{a^2+c^2-b^2}{2a}, \frac{2\Delta}{a}, 0\right)$ &

$$D\left(\frac{a^2 + b^2 - c^2}{2a}, \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}, \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}}\right)$$

Where, Δ is the area of each acute triangular face with sides a, b & c of the disphenoid.

The vertical height of disphenoid (isosceles tetrahedron): The disphenoid has four congruent acute-triangular faces thus by symmetry, the vertical height of each vertex from its opposite face (base) is equal & it is equal to the vertical height DE of vertex D from triangular face ABC lying on the XY-plane

$$DE = z \text{ coordinate of vertex D} = \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}}$$

The volume of disphenoid (isosceles tetrahedron): The volume (V) of disphenoid ABCD (see figure 2 above) is given as

$$\begin{aligned} V &= \frac{1}{3}(\text{Area of triangular base ABC})(\text{normal height of vertex D from base ABC}) = \frac{1}{3}(\Delta)(DE) \\ &= \frac{1}{3}(\Delta) \left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}} \right) \\ &= \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{6\sqrt{2}} \end{aligned}$$

$$\text{Volume of disphenoid, } V = \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{72}$$

Above is the general formula to compute the volume of a disphenoid having four congruent faces each as an acute-angled triangle with sides a, b & c

Now, the vertical height (H) of disphenoid in terms of volume (V) & area of face (Δ) is given as

$$H = \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}} = \frac{3\sqrt{\frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{72}}}{\Delta} = \frac{3V}{\Delta}$$

The surface area of disphenoid (isosceles tetrahedron): The disphenoid consists of four congruent triangular faces hence the surface area of disphenoid

$$A_s = 4(\text{Area of acute triangular face ABC}) = 4\Delta = 4\sqrt{s(s-a)(s-b)(s-c)}$$

Where, $s = \frac{a+b+c}{2}$ is the semi-perimeter of any of four congruent acute-triangular faces of a disphenoid

The radius of sphere inscribed by the disphenoid (isosceles tetrahedron): Let r be the radius of sphere

inscribed by the disphenoid ABCD, the centre O of inscribed sphere is at an equal normal distance r from all four congruent triangular faces (As shown in the figure 3) Drop the perpendiculars from in-centre O to four faces to get four congruent elementary tetrahedrons each of base area Δ & normal height r . If we add the volumes of these four congruent elementary tetrahedrons then we get the volume of original disphenoid thus the volume of disphenoid

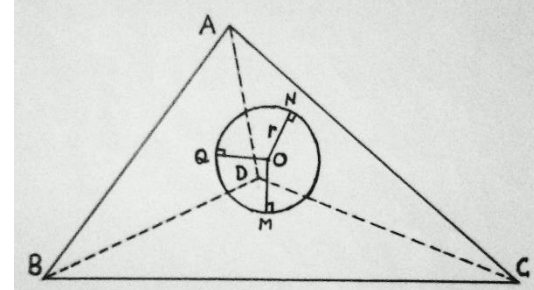


Figure 3: The centre O of inscribed sphere is at an equal normal distance r from all four congruent acute-triangular faces of disphenoid ABCD.

$$V = 4(\text{volume of elementary tetrahedron})$$

$$V = 4 \left(\frac{1}{3} (\Delta)(r) \right) \Rightarrow r = \frac{3V}{4\Delta}$$

Inradius of disphenoid, $r = \frac{3V}{4\Delta}$

The radius of sphere circumscribing the disphenoid (isosceles tetrahedron): Let R be the radius of sphere circumscribing the disphenoid

ABCD, the centre P of the circumscribed sphere is at an equal distance R from all four vertices A, B, C & D (As shown in the figure 4). Let the coordinates of circum-centre P be (h, k, l) in 3D space.

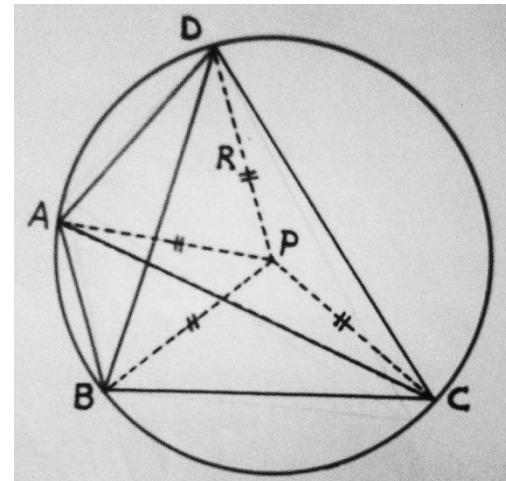


Figure 4: The centre P of circumscribed sphere is at an equal distance R from all four vertices A, B, C & D of disphenoid ABCD.

Now, using distance formula, the distance between the circum-centre $P(h, k, l)$ & the vertex $A(0, 0, 0)$ of disphenoid ABCD is given as

$$PA = \sqrt{(h-0)^2 + (k-0)^2 + (l-0)^2} = R$$

$$h^2 + k^2 + l^2 = R^2 \quad \dots \dots \dots (5)$$

Similarly, the distance between $P(h, k, l)$ & $B(a, 0, 0)$ is given as

$$PB = \sqrt{(h-a)^2 + (k-0)^2 + (l-0)^2} = R$$

$$(h-a)^2 + (k-0)^2 + (l-0)^2 = R^2$$

$$h^2 + k^2 + l^2 - 2ah + a^2 = R^2$$

Setting the value of $h^2 + k^2 + l^2$ from (5),

$$R^2 - 2ah + a^2 = R^2$$

$$h = \frac{a}{2} \quad \dots \dots \dots (6)$$

Similarly, the distance between $P(h, k, l)$ & $C\left(\frac{a^2+c^2-b^2}{2a}, \frac{2\Delta}{a}, 0\right)$ is given as

$$PC = \sqrt{\left(h - \frac{a^2 + c^2 - b^2}{2a}\right)^2 + \left(k - \frac{2\Delta}{a}\right)^2 + (l-0)^2} = R$$

$$\left(h - \frac{a^2 + c^2 - b^2}{2a}\right)^2 + \left(k - \frac{2\Delta}{a}\right)^2 + (l - 0)^2 = R^2$$

$$h^2 + k^2 + l^2 - 2\left(\frac{a^2 + c^2 - b^2}{2a}\right)h + \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 - 2\left(\frac{2\Delta}{a}\right)k + \left(\frac{2\Delta}{a}\right)^2 = R^2$$

Setting the value of $h^2 + k^2 + l^2$ from (5) & the value of h from (6),

$$R^2 - 2\left(\frac{a^2 + c^2 - b^2}{2a}\right)\left(\frac{a}{2}\right) + \left(\frac{a^2 + c^2 - b^2}{2a}\right)^2 - \left(\frac{4\Delta}{a}\right)k + \left(\frac{2\Delta}{a}\right)^2 = R^2$$

$$\begin{aligned} \left(\frac{4\Delta}{a}\right)k &= -\frac{(a^2 + c^2 - b^2)}{2} + \frac{(a^2 + c^2 - b^2)^2}{4a^2} + \frac{4\Delta^2}{a^2} \\ &= \frac{-2a^2(a^2 + c^2 - b^2) + (a^2 + c^2 - b^2)^2 + 16\Delta^2}{4a^2} \\ &= \frac{(a^2 + c^2 - b^2)(a^2 + c^2 - b^2 - 2a^2) + 16\Delta^2}{4a^2} \\ &= \frac{b^4 + c^4 - a^4 - 2b^2c^2 + 16\Delta^2}{4a^2} \end{aligned}$$

But from Heron's formula, $\Delta = \sqrt{s(s-a)(s-b)(s-c)} \Rightarrow 16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$

Setting the value of $16\Delta^2$ in above equation, we should get

$$\begin{aligned} \left(\frac{4\Delta}{a}\right)k &= \frac{b^4 + c^4 - a^4 - 2b^2c^2 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4}{4a^2} \\ k &= \frac{b^4 + c^4 - a^4 - 2b^2c^2 + 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4}{16a\Delta} \\ k &= \frac{2a^2b^2 + 2a^2c^2 - 2a^4}{16a\Delta} \\ k &= \frac{ab^2 + ac^2 - a^3}{8\Delta} \dots \dots \dots (7) \end{aligned}$$

Similarly, the distance between $P(h, k, l)$ & $D\left(\frac{a^2+b^2-c^2}{2a}, \frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}, \frac{\sqrt{(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)}}{2\Delta\sqrt{2}}\right)$ is given as

$$\begin{aligned} PD &= \sqrt{\left(h - \frac{a^2 + b^2 - c^2}{2a}\right)^2 + \left(k - \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}\right)^2 + \left(l - \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}}\right)^2} = R \\ &\left(h - \frac{a^2 + b^2 - c^2}{2a}\right)^2 + \left(k - \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}\right)^2 + \left(l - \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}}\right)^2 = R^2 \\ h^2 + k^2 + l^2 &+ \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2 - 2\left(\frac{a^2 + b^2 - c^2}{2a}\right)h + \left(\frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}\right)^2 - 2\left(\frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}\right)k \\ &+ \left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}}\right)^2 - 2\left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}}\right)l = R^2 \end{aligned}$$

Setting the value of $h^2 + k^2 + l^2$ from (5), the value of h from (6) & the value of k from (7),

$$\begin{aligned}
 R^2 + \frac{(a^2 + b^2 - c^2)^2}{4a^2} - 2 \left(\frac{a^2 + b^2 - c^2}{2a} \right) \left(\frac{a}{2} \right) + \frac{(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}{64a^2\Delta^2} \\
 - 2 \left(\frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta} \right) \left(\frac{ab^2 + ac^2 - a^3}{8\Delta} \right) + \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{8\Delta^2} \\
 - 2 \left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{2\Delta\sqrt{2}} \right) l = R^2 \\
 \\
 \left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{\Delta\sqrt{2}} \right) l \\
 = \frac{(a^2 + b^2 - c^2)^2}{4a^2} - \frac{a^2 + b^2 - c^2}{2} + \frac{(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}{64a^2\Delta^2} \\
 - \frac{2(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(ab^2 + ac^2 - a^3)}{64a\Delta^2} + \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{8\Delta^2} \\
 \\
 = \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)}{4a^2} + \frac{(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}{64a\Delta^2} - \frac{2(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(ab^2 + ac^2 - a^3)}{64a\Delta^2} \\
 + \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{8\Delta^2} \\
 \\
 = \frac{16\Delta^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)}{64a^2\Delta^2} + \frac{(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2}{64a^2\Delta^2} - \frac{2a(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(ab^2 + ac^2 - a^3)}{64a^2\Delta^2} \\
 + \frac{8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{64a^2\Delta^2} \\
 \\
 64a^2\Delta^2 \left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{\Delta\sqrt{2}} \right) l \\
 = 16\Delta^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2) + (2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2 \\
 - 2a(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(ab^2 + ac^2 - a^3) + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 (32\sqrt{2}a^2\Delta\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}) l \\
 = 16\Delta^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2) + (2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)^2 \\
 - 2a(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(ab^2 + ac^2 - a^3) + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = 16\Delta^2(b^4 + c^4 - 2b^2c^2 - a^4) + (2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 + 2a^4) \\
 + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = 16\Delta^2(b^4 + c^4 - 2b^2c^2 - a^4) + (2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4)(b^4 + c^4 - 2b^2c^2 - a^4) + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = (b^4 + c^4 - 2b^2c^2 - a^4)(16\Delta^2 + 2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4) + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = (b^4 + c^4 - 2b^2c^2 - a^4)(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 + 2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4) \\
 + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = (b^4 + c^4 - 2b^2c^2 - a^4)(4a^2b^2 + 4a^2c^2 - 4a^4) + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = 4a^2((b^2 - c^2)^2 - a^4)(b^2 + c^2 - a^2) + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = 4a^2(b^2 - c^2 + a^2)(b^2 - c^2 - a^2)(b^2 + c^2 - a^2) + 8a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 = 4a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(b^2 - c^2 - a^2 + 2a^2) \\
 \\
 (32\sqrt{2}a^2\Delta\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}) l = 4a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2) \\
 \\
 l = \frac{4a^2(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{32\sqrt{2}a^2\Delta\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}} \\
 \\
 l = \frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{8\sqrt{2}\Delta} \\
 \\
 l = \frac{3\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{4\Delta} \\
 \\
 l = \frac{3V}{4\Delta} = r = \text{Radius of inscribed sphere} \quad \dots \dots \dots (8)
 \end{aligned}$$

The above result (eq(8)) shows that

1. Distance of circum-centre P from the acute-triangular face ABC (lying on the XY-plane) is equal to the radius r of inscribed sphere of disphenoid ABCD
2. The perpendicular drawn from the circum-centre P of disphenoid ABCD falls at the circum-centre of acute-triangular face ABC
3. By symmetry of disphenoid that four faces are acute angled triangles, the circum-centre of disphenoid is at an equal normal distance from each acute-triangular face.
4. The normal distance of circum-centre from each face is equal to the in-radius of disphenoid, hence by symmetry the circum-centre must be coincident with the in-centre of a disphenoid.
5. Inscribed sphere touches each of four congruent acute-triangular faces at circum-centre of that face

Now, substituting the values of h, k & l from (6), (7) & (8) respectively into (5), the circum-radius R is given as

$$\begin{aligned}
 R &= \sqrt{h^2 + k^2 + l^2} \\
 &= \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{ab^2 + ac^2 - a^3}{8\Delta}\right)^2 + \left(\frac{\sqrt{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}}{8\sqrt{2}\Delta}\right)^2} \\
 &= \sqrt{\frac{a^2}{4} + \frac{(ab^2 + ac^2 - a^3)^2}{64\Delta^2} + \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{128\Delta^2}} \\
 &= \sqrt{\frac{32a^2\Delta^2 + 2(ab^2 + ac^2 - a^3)^2 + (a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{128\Delta^2}} \\
 &= \sqrt{\frac{32a^2\Delta^2 + 2a^2(b^2 + c^2 - a^2)^2 + (b^2 + c^2 - a^2)(a^4 - b^4 - c^4 + 2b^2c^2)}{128\Delta^2}} \\
 &= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(2a^2(b^2 + c^2 - a^2) + a^4 - b^4 - c^4 + 2b^2c^2)}{128\Delta^2}} \\
 &= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(2a^2b^2 + 2a^2c^2 - 2a^4 + a^4 - b^4 - c^4 + 2b^2c^2)}{128\Delta^2}} \\
 &= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4)}{128\Delta^2}} \\
 &= \sqrt{\frac{32a^2\Delta^2 + (b^2 + c^2 - a^2)(16\Delta^2)}{128\Delta^2}} \quad (\text{since, } 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 = 16\Delta^2) \\
 &= \sqrt{\frac{2a^2 + b^2 + c^2 - a^2}{8}} = \sqrt{\frac{a^2 + b^2 + c^2}{8}}
 \end{aligned}$$

For a disphenoid (isosceles tetrahedron) having three unequal edges a, b & c , radius of circumscribed sphere

$$\text{Circumradius of disphenoid, } R = \sqrt{\frac{a^2 + b^2 + c^2}{8}}$$

The coordinates of circum-centre or in-centre of disphenoid (isosceles tetrahedron): The coordinates of in-centre or circum-centre $P(h, k, l)$ of a disphenoid ABCD having vertices (for the optimal configuration in 3D space) $A(0, 0, 0), B(a, 0, 0), C\left(\frac{a^2+c^2-b^2}{2a}, \frac{2\Delta}{a}, 0\right)$ & $D\left(\frac{a^2+b^2-c^2}{2a}, \frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}, \frac{3V}{\Delta}\right)$ is given by substituting the values of h, k & l from (6), (7) & (8) respectively as follows

$$\text{Incentre or Circumcentre of disphenoid, } P(h, k, l) \equiv \left(\frac{a}{2}, \frac{ab^2 + ac^2 - a^3}{8\Delta}, \frac{3V}{4\Delta}\right)$$

Where, V is the volume of disphenoid & Δ is the area of each acute-angled triangular face with sides a, b & c

The in-dentre & the circum-centre of a disphenoid (isosceles tetrahedron) are coincident

The coordinates of centroid of disphenoid (isosceles tetrahedron): The coordinates of centroid $G(\bar{x}, \bar{y}, \bar{z})$ of a disphenoid ABCD having vertices (for the optimal configuration in 3D space) $A(0, 0, 0), B(a, 0, 0), C\left(\frac{a^2+c^2-b^2}{2a}, \frac{2\Delta}{a}, 0\right)$ & $D\left(\frac{a^2+b^2-c^2}{2a}, \frac{2a^2b^2-2b^2c^2+2a^2c^2-3a^4+b^4+c^4}{8a\Delta}, \frac{3V}{\Delta}\right)$ is given as follows

$$G(\bar{x}, \bar{y}, \bar{z}) \equiv \left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

$$\bar{x} = \frac{0 + a + \frac{a^2 + c^2 - b^2}{2a} + \frac{a^2 + b^2 - c^2}{2a}}{4} = \frac{2a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2}{8a} = \frac{4a^2}{8a} = \frac{a}{2}$$

$$\bar{y} = \frac{0 + 0 + \frac{2\Delta}{a} + \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}}{4}$$

$$\bar{y} = \frac{16\Delta^2 + 2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{32a\Delta}$$

But from Heron's formula, $\Delta = \sqrt{s(s-a)(s-b)(s-c)} \Rightarrow 16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$
Setting the value of $16\Delta^2$ in above equation, we should get

$$\bar{y} = \frac{2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4 + 2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{32a\Delta}$$

$$\bar{y} = \frac{4a^2b^2 + 4a^2c^2 - 4a^4}{32a\Delta} = \frac{ab^2 + ac^2 - a^3}{8\Delta}$$

$$\bar{z} = \frac{0 + 0 + 0 + \frac{3V}{\Delta}}{4} = \frac{3V}{4\Delta}$$

$$\text{Centroid of disphenoid, } G(\bar{x}, \bar{y}, \bar{z}) \equiv \left(\frac{a}{2}, \frac{ab^2 + ac^2 - a^3}{8\Delta}, \frac{3V}{4\Delta}\right)$$

The above result shows that the coordinates of centroid $G(\bar{x}, \bar{y}, \bar{z})$ are same as that of in-centre or circum-centre of a disphenoid hence we can conclude that **in-centre, circum-centre & centroid of a disphenoid (isosceles tetrahedron with four congruent faces each as an acute-angled triangle) are always coincident.**

The solid angle subtended by disphenoid (isosceles tetrahedron) at its vertex: All four acute-angled triangular faces of disphenoid are congruent hence the solid angle subtended by each face at its opposite vertex will be equal to the solid angle subtended by disphenoid at any of its four vertices. The **solid angle subtended by any tetrahedron (or disphenoid) at its vertex** is given by the general formula

$$\omega = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) - \sin^{-1}\left(\frac{\cos\beta - \cos\gamma\cos\alpha}{\sin\gamma\sin\alpha}\right) - \sin^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right)$$

Where, α, β & γ are the angles between consecutive lateral edges meeting at the concerned vertex of disphenoid. The angles α, β & γ meeting at any vertex of a disphenoid can be easily computed by using **cosine formula** as follows

$$\alpha = \cos^{-1}\left(\frac{b^2 + c^2 - a^2}{2bc}\right), \quad \beta = \cos^{-1}\left(\frac{a^2 + c^2 - b^2}{2ac}\right), \quad \gamma = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$

$$(\forall \mathbf{0} < \alpha, \beta, \gamma < \pi \text{ \& } \alpha + \beta + \gamma < 2\pi)$$

Formula (Governing equation) of disphenoid: Consider a disphenoid (isosceles tetrahedron) having three unequal edges a, b, c meeting at the same vertex. Now, the area Δ of each of four congruent acute-triangular faces with sides a, b & c is given by Heron's formula as follows

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)} \quad \forall s = \frac{a+b+c}{2}$$

$$\Delta^2 = s(s-a)(s-b)(s-c)$$

$$\Delta^2 = \left(\frac{a+b+c}{2}\right)\left(\frac{a+b+c}{2} - a\right)\left(\frac{a+b+c}{2} - b\right)\left(\frac{a+b+c}{2} - c\right)$$

$$\Delta^2 = \frac{(a+b+c)(b+c-a)(a+c-b)(a+b-c)}{16}$$

$$16\Delta^2 = \{(b+c+a)(b+c-a)\}\{(a-(b-c))(a+b-c)\}$$

$$16\Delta^2 = \{(b+c)^2 - a^2\}\{a^2 - (b-c)^2\}$$

$$16\Delta^2 = \{b^2 + c^2 + 2bc - a^2\}\{a^2 - b^2 - c^2 + 2bc\}$$

$$16\Delta^2 = \{2bc + (b^2 + c^2 - a^2)\}\{2bc - (b^2 + c^2 - a^2)\}$$

$$16\Delta^2 = (2bc)^2 - (b^2 + c^2 - a^2)^2$$

$$16\Delta^2 = 4b^2c^2 - b^4 - c^4 - a^4 - 2b^2c^2 + 2a^2c^2 + 2a^2b^2$$

$$16\Delta^2 = 2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4$$

$$16\Delta^2 R^2 = (2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4)R^2 \quad (\text{multiplying } R^2 \text{ on both sides})$$

$$16\Delta^2 R^2 = (2a^2b^2 + 2b^2c^2 + 2a^2c^2 - a^4 - b^4 - c^4) \left(\frac{a^2 + b^2 + c^2}{8}\right) \quad (\text{setting value of } R^2)$$

$$16\Delta^2 R^2$$

$$= \frac{2a^4b^2 + 2a^2b^2c^2 + 2a^4c^2 - a^6 - a^2b^4 - a^2c^4 + 2a^2b^4 + 2b^4c^2 + 2a^2b^2c^2 - a^4b^2 - b^6 - b^2c^4 + 2a^2b^2c^2 + 2b^2c^4 + 2a^2c^4 - a^4c^2 - b^4c^2 - c^6}{8}$$

$$16\Delta^2 R^2 = \frac{6a^2b^2c^2 + a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4 - a^6 - b^6 - c^6}{8}$$

$$16\Delta^2 R^2 = \frac{8a^2b^2c^2 + (a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4 - a^6 - b^6 - c^6 - 2a^2b^2c^2)}{8}$$

$$16\Delta^2 R^2 = a^2b^2c^2 + \frac{a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4 - a^6 - b^6 - c^6 - 2a^2b^2c^2}{8}$$

$$16\Delta^2 R^2 = a^2b^2c^2 + \frac{(a^2 + b^2 - c^2)(2a^2b^2 + c^4 - a^4 - b^4)}{8}$$

$$16\Delta^2 R^2 = a^2b^2c^2 + \frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{8}$$

$$16\Delta^2 R^2 = a^2b^2c^2 + 9\left(\frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{72}\right)$$

$$16\Delta^2 R^2 = a^2b^2c^2 + 9(V^2) \quad \text{(substituting by } V^2\text{)}$$

$$\mathbf{16\Delta^2 R^2 = a^2b^2c^2 + 9V^2}$$

Above relation (called **formula of disphenoid**) is always satisfied by a disphenoid (isosceles tetrahedron) & also very important for directly computing the volume V or circum-radius R when other parameters a, b & c are known

Conclusion: Let there be a disphenoid (isosceles tetrahedron) having four congruent faces each as an acute-angled triangle with sides a, b & c then all the important parameters of disphenoid can be computed as tabulated below

Volume	$V = \sqrt{\frac{(a^2 + b^2 - c^2)(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{72}}$
Surface area	$4\Delta = 4\sqrt{s(s-a)(s-b)(s-c)}$
Vertical height	$\frac{3V}{\Delta}$
In-radius (radius of inscribed sphere)	$\frac{3V}{4\Delta}$
circum-radius (radius of circumscribed sphere)	$\sqrt{\frac{a^2 + b^2 + c^2}{8}}$
Coordinates of four vertices (Optimal configuration of a disphenoid in 3D space)	$(0, 0, 0), (a, 0, 0), \left(\frac{a^2 + c^2 - b^2}{2a}, \frac{2\Delta}{a}, 0\right) \&$ $\left(\frac{a^2 + b^2 - c^2}{2a}, \frac{2a^2b^2 - 2b^2c^2 + 2a^2c^2 - 3a^4 + b^4 + c^4}{8a\Delta}, \frac{3V}{\Delta}\right)$

Coordinates of coincident in-centre, circum-centre & centroid	$\left(\frac{a}{2}, \frac{ab^2 + ac^2 - a^3}{8\Delta}, \frac{3V}{4\Delta}\right)$
<p>A disphenoid (isosceles tetrahedron) always satisfies the relation: $16\Delta^2 R^2 = a^2 b^2 c^2 + 9V^2$</p> <p>Where, V is the volume, R is the radius of circumscribed sphere & Δ is the area of each acute-angled triangular face with sides a, b, c in a disphenoid</p>	

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