Solid angle subtended by a regular n-gonal right pyramid (solid or hollow) at its apex

(Application of HCR's Theory of Polygon)

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Introduction: Here we are to derive the formula for finding out the solid angle subtended by a regular n-gonal right pyramid at its apex by using formula of solid angle subtended by a regular n-polygon at any point lying on the perpendicular passing through its centre which has already been derived in **HCR's Theory of Polygon**. The solid angle subtended by a regular n-gonal right pyramid will be derived in terms of apex angle α (i.e. angle between any two consecutive lateral edges meeting at the apex) & the number n of sides in regular polygonal base of the right pyramid.

Derivation: Let there be a right pyramid (solid or hollow) with normal height 'H', apex point 'P', angle between any two consecutive lateral edges 'α' & base as a regular polygon with 'n' no. of the sides each of length 'a' (as shown in the figure-1), the upper part shows the isometric front view & lower one the top view)

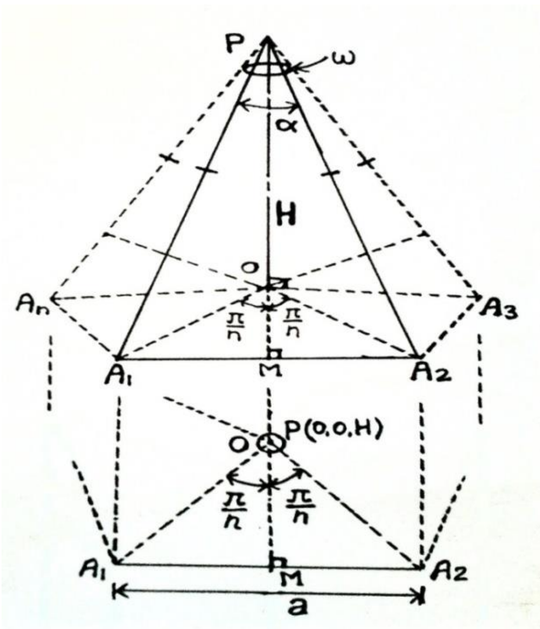

Figure 1: A perpendicular PM is dropped from apex P to the mid-point M of side A_1A_2 of base of regular n-gonal right pyramid to obtain a right ΔPMA_1

Now, join centre 'O' of the base to all the vertices $A_1, A_2, A_3 \dots\dots\dots.\ A_n$ of the regular polygonal base to obtain n number of congruent isosceles triangles (As shown in figure-1).

Consider an isosceles ΔA_1OA_2 & drop the perpendicular OM to the mid-point 'M' of side A_1A_2 of regular polygonal base to obtain two congruent right triangles ΔA_1MO & ΔA_2MO

In right ΔA_1MO (see lower part in fig-1)

$$\tan\angle A_1OM = \frac{A_1M}{OM}$$

$$\tan\frac{\pi}{n} = \frac{\left(\frac{a}{2}\right)}{\text{OM}}$$

$$\text{OM} = \frac{a}{2}\cot\frac{\pi}{n}$$

In right ΔPMA_1 (see upper part in above fig-1)

$$\tan\angle A_1PM = \frac{A_1M}{PM}$$

$$\tan\frac{\alpha}{2} = \frac{\left(\frac{a}{2}\right)}{\text{PM}}$$

$$\text{PM} = \frac{a}{2}\cot\frac{\alpha}{2}$$

By joining the point ‘P’ to the centre ‘O’ and the point ‘M’, we obtain a right ΔPOM (as shown in figure-2)

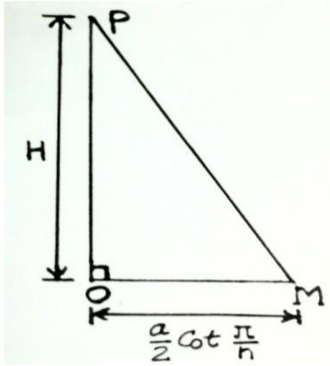

Figure 2: $PO = H$ **is the normal height of n-gonal right pyramid**

In right ΔPOM

$$PM^2 = OP^2 + OM^2$$

$$\left(\frac{a}{2}\cot\frac{\alpha}{2}\right)^2 = H^2 + \left(\frac{a}{2}\cot\frac{\pi}{n}\right)^2$$

$$H^2 = \frac{a^2}{4}\left(\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}\right)$$

$$\boldsymbol{H = \frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}}}$$

Above is the general formula to compute the normal height of a regular n-gonal right pyramid when apex angle α , number of sides n & length of each side a of regular polygonal base are known.

The solid angle ($\omega_{pyramid}$) subtended by the regular n-gonal right pyramid at its apex P will be equal to the solid angle ($\omega_{polygon}$) subtended by regular n-gon $A_1A_2A_3 \dots\dots A_n$ of each side a, at the apex P lying a normal height H from the centre ‘O’ which is given by the formula of **HCR’s Theory of Polygon** as follows

$$\boldsymbol{\omega_{polygon} = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right)} \quad \forall\,(n \in N\ \&\ n \geq 3)$$

Now, setting the value of H in the above general formula of solid angle, we get the solid angle subtended by the regular n-gonal right pyramid at its apex

$$\omega_{pyramid} = 2\pi - 2n\sin^{-1}\left(\frac{2\left(\frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}}\right)\sin\frac{\pi}{n}}{\sqrt{4\left(\frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}}\right)^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{a\sin\frac{\pi}{n}\sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}}}{\sqrt{a^2\cot^2\frac{\alpha}{2} - a^2\cot^2\frac{\pi}{n} + a^2\cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{a\sin\frac{\pi}{n}\sqrt{\frac{1}{\tan^2\frac{\alpha}{2}} - \frac{1}{\tan^2\frac{\pi}{n}}}}{a\cot\frac{\alpha}{2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{a\sin\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}}{a\cot\frac{\alpha}{2}\tan\frac{\alpha}{2}\tan\frac{\pi}{n}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sin\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}}{\left(\frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}}\right)}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)$$

Hence, the **solid angle (ω) subtended at the apex by any right pyramid with normal height H, base as a regular polygon with n number of sides each of length a and the apex angle α (i.e. angle between any two consecutive lateral edges)**, is given by the following formula

$$\boldsymbol{\omega = 2\pi - 2n\sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{\sqrt{4H^2 + a^2\cot^2\frac{\pi}{n}}}\right) = 2\pi - 2n\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{\tan^2\frac{\pi}{n} - \tan^2\frac{\alpha}{2}}\right)}$$

$$\text{Where,}\quad n \in N,\ n \geq 3\ \&\ 0 \leq \alpha \leq \frac{2\pi}{n}$$

And the relation among the apex angle 'α', side of regular n-gonal base a & normal height 'H' of the right pyramid, is given by the above equation (as highlighted in green colour above) as follows

$$\boldsymbol{H = \frac{a}{2}\sqrt{\cot^2\frac{\alpha}{2} - \cot^2\frac{\pi}{n}}}$$

Thus, any of above formula can be used to compute the solid angle subtended by a regular n-gonal right pyramid at its vertex when

1) $H =$ normal height, $a =$ length of each side & $n =$ number of sides of regular polygonal base are known or
2) $\alpha =$ apex angle & $n =$ number of sides of regular polygonal base of right pyramid are known.

Note: Above articles had been *derived & illustrated* by **Mr H.C. Rajpoot** (**M Tech, Production Engineering**)

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