# Regular N-gonal Right Antiprism: Application of HCR's Theory of Polygon

*Harish Chandra Rajpoot*

*Department of Mechanical Engineering, Indian Institute of Technology Bombay, Mumbai, India-400076*

Email: hcrajpoot@iitb.ac.in Date: 23-05-2023

## Abstract

A regular n-gonal right antiprism is a semiregular convex polyhedron which has 2n identical vertices all lying on a sphere, 4n edges, and (2n+2) faces out of which 2 are congruent regular n-sided polygons, and 2n are congruent equilateral triangles such that all the faces have equal side. The equilateral triangular faces meet the regular polygonal faces at the common edges and vertices alternatively such that three equilateral triangular faces meet at each of 2n vertices. This paper presents, in details, the mathematical derivations of the generalized and analytic formula which are used to determine the different important parameters in terms of edge length, such as normal distances of faces, normal height, radius of circumscribed sphere, surface area, volume, dihedral angles between adjacent faces, solid angle subtended by each face at the centre, and solid angle subtended by polygonal antiprism at each of its 2n vertices using HCR's Theory of Polygon. All the generalized formulae have been derived using simple trigonometry, and 2D geometry which are difficult to derive using any other methods.

***Keywords***: Regular n-gonal right antiprism, generalized formula of antiprism, solid angles, dihedral angles

## 1. Introduction

A regular n-gonal right antiprism has 2n congruent equilateral triangular faces, 2 identical parallel regular n-gonal base faces, 2n identical vertices all lying on a sphere, and 4n edges such that two opposite and parallel regular polygons are relatively rotated through an angle of $\pi/n$ about the axis passing through their centres and perpendicular to their planes. It is a convex polyhedron (i.e. internal angle between any two adjacent faces is less than $\pi$) which has each of its 2n+2 faces as a regular polygon and all its 2n vertices identical therefore it is called a semiregular polyhedron. It is also known as uniform n-antiprism, uniform, equilateral antiprism [1]. (as shown in the Figure-1 below).

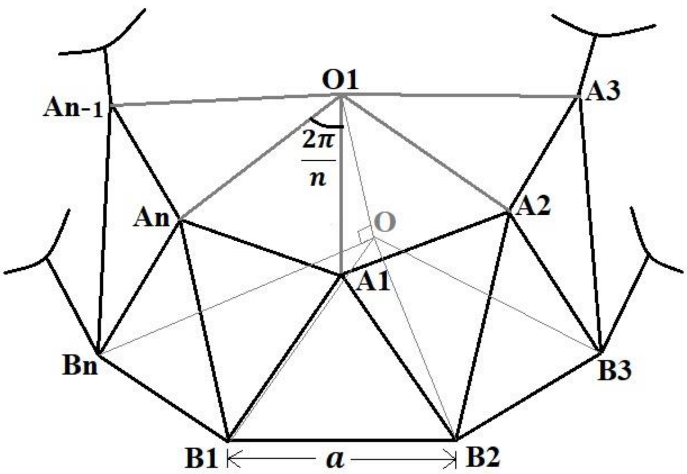


**Figure-1: A regular n-gonal right antiprism consists of two identical, opposite and parallel regular polygons each with n no. of sides separated by a band of 2n congruent equilateral triangles, 2n identical vertices lying on a sphere and 4n edges. All 2n+2 faces of the antiprism have a side $a$.**

Depending on the number of sides of regular n-gonal face, a regular polygonal right antiprism has various geometric shapes which form an infinite family of antiprism. For $n = 3$, the antiprism has its simplest form having $2n + 2 = 8$ equilateral triangular faces which is called triangular antiprism. It's also a regular tetrahedron [2]. (as shown in the Figure-2 below).

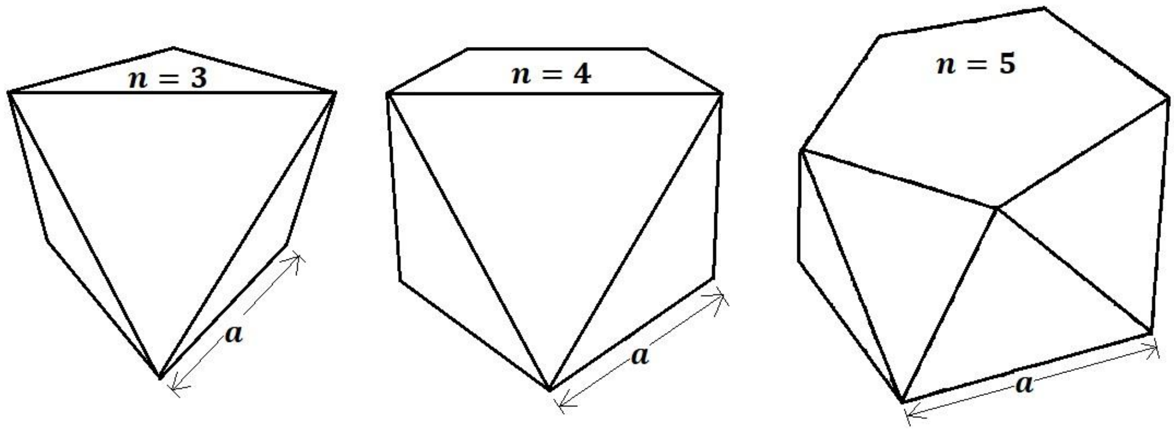


**Figure-2: The various geometric shapes of a regular polygonal right antiprism form an infinite family. a) triangular right antiprism, b) square right antiprism, and c) pentagonal right antiprism**

## 2. Derivation of parameters

Let's consider a regular n-gonal right antiprism with edge length $a$ (as shown in the Figure-2) such that

$H =$ Normal height i.e. the normal distance between the regular n-sided polygonal faces of the right antiprism

$H_n =$ Normal distance of each regular polygonal face from the centre of the right antiprism

$H_T =$ Normal distance of each of 2n congruent regular triangular faces from the centre of the right antiprism

$R_o =$ Radius of circumscribed sphere i.e. distance of each of 2n identical vertices from the centre of the antiprism

$A_s =$ Surface area of the antiprism

$V =$ Volume of the antiprism

$\theta_{TTE} =$ Dihedral angle between any two adjacent equilateral triangular faces having a common edge

$\theta_{TTV} =$ Dihedral angle between equilateral triangular and regular polygonal faces having a common vertex

$\theta_{TPE} =$ Dihedral angle between equilateral triangular and regular polygonal faces having a common edge

$\theta_{TPV} =$ Dihedral angle between equilateral triangular and regular polygonal faces having a common vertex

$\omega_T =$ Solid angle subtended by each regular triangular face at the centre of the right antiprism

$\omega_n =$ Solid angle subtended by each regular n-sided polygonal face at the centre of the right antiprism

$\omega_V =$ Solid angle subtended by the right antiprism at each of its 2n identical vertices

### 2.1. Normal distance of regular n-sided polygonal face from the centre

Let $H_n$ be the normal distance of regular n-sided polygonal face from the centre O of the polygonal antiprism having edge length $a$. Now, the circum-radius of regular polygon $A_1A_2A_3 \dots \dots A_{n-1}A_n$ with centre $O_1$ (see the above Figure-2) is given as

$$\sin\frac{\pi}{n} = \frac{MA_1}{O_1A_1} \;\Rightarrow\; O_1A_1 = \frac{MA_1}{\sin\frac{\pi}{n}} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

In right $\Delta OO_1A_1$ (see the above Figure-2), applying Pythagorean theorem as follows

$$OO_1 = \sqrt{(OA_1)^2 - (O_1A_1)^2} = \sqrt{(R_o)^2 - \left(\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\right)^2}$$

$$\Rightarrow\; H_n = \sqrt{{R_o}^2 - \frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}} \qquad \dots\dots\dots\dots\dots\dots \;(1)$$

### 2.2. The solid angle subtended by regular n-sided polygonal face at the centre

The solid angle subtended by any polygonal plane, having $n$ no. of sides each of length $a$, at any point lying on the perpendicular axis passing through the centre of polygon at a distance $h$ is given by the generalized formula from HCR's Theory of Polygon [1] as follows

$$\omega = 2\pi - 2n\sin^{-1}\left(\frac{2h\sin\frac{\pi}{n}}{\sqrt{4h^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

Now, substituting the corresponding value i.e. $h = H_n =$ normal distance of regular polygonal face from the centre O of the antiprism (as shown in the above Figure-2) in the above generalized formula, the solid angle $\omega_n$ subtended by each regular polygonal face at the centre O is obtained as follows

$$\omega_n = 2\pi - 2n\sin^{-1}\left(\frac{2H_n\sin\frac{\pi}{n}}{\sqrt{4{H_n}^2 + a^2\cot^2\frac{\pi}{n}}}\right) = 2\pi - 2n\sin^{-1}\left(\frac{2\sqrt{{R_o}^2 - \frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}}\,\sin\frac{\pi}{n}}{\sqrt{4\left(\sqrt{{R_o}^2 - \frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}}\right)^2 + a^2\cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sqrt{4{R_o}^2\sin^2\frac{\pi}{n} - 4\cdot\frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}\sin^2\frac{\pi}{n}}}{\sqrt{4{R_o}^2 - 4\cdot\frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n} + a^2\cot^2\frac{\pi}{n}}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\frac{\sqrt{4{R_o}^2\sin^2\frac{\pi}{n} - a^2}}{\sqrt{4{R_o}^2 - a^2\left(\operatorname{cosec}^2\frac{\pi}{n} - \cot^2\frac{\pi}{n}\right)}}\right) = 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4{R_o}^2\sin^2\frac{\pi}{n} - a^2}{4{R_o}^2 - a^2}}\right)$$

$$= 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4\left(\frac{R_o}{a}\right)^2\sin^2\frac{\pi}{n} - 1}{4\left(\frac{R_o}{a}\right)^2 - 1}}\right) = 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\right)$$

$$\omega_n = 2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\right) \quad \dots\dots\dots\dots\dots (2)$$

Where, $x = R_o/a$ is some unknown ratio.

### 2.3. Normal distance of equilateral triangular face from the centre

Let $H_T$ be the normal distance of equilateral triangular face from the centre O of the polygonal antiprism having edge length $a$. Now, the circum-radius of regular triangle with centre $O_2$ (see the above Figure-2) is given as

$$\sin\frac{\pi}{3} = \frac{MA_1}{O_2A_1} \quad \Rightarrow \quad O_2A_1 = \frac{MA_1}{\sin\frac{\pi}{3}} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{3} = \frac{a}{2}\cdot\frac{2}{\sqrt{3}} = \frac{a}{\sqrt{3}}$$

In right $\Delta OO_2A_1$ (see the above Figure-2), applying Pythagorean theorem as follows

$$OO_2 = \sqrt{(OA_1)^2 - (O_2A_1)^2} = \sqrt{(R_o)^2 - \left(\frac{a}{\sqrt{3}}\right)^2}$$

$$\Rightarrow \ H_T = \sqrt{{R_o}^2 - \frac{a^2}{3}} \quad \dots\dots\dots\dots\dots\dots \ (3)$$

### 2.4. The solid angle subtended by equilateral triangular face at the centre

Similarly, substituting the corresponding value i.e. $h = H_T =$ normal distance of equilateral triangular face ($n = 3$) from the centre O of the antiprism (as shown in the above Figure-2) in the above generalized formula, the solid angle $\omega_T$ subtended by each equilateral triangular face at the centre O is obtained as follows

$$\omega_T = 2\pi - 2(3)\sin^{-1}\left(\frac{2H_T\sin\frac{\pi}{3}}{\sqrt{4{H_T}^2 + a^2\cot^2\frac{\pi}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{2\sqrt{{R_o}^2 - \frac{a^2}{3}}\,\frac{\sqrt{3}}{2}}{\sqrt{4\left(\sqrt{{R_o}^2 - \frac{a^2}{3}}\right)^2 + a^2\left(\frac{1}{\sqrt{3}}\right)^2}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{\sqrt{3}\sqrt{{R_o}^2 - \frac{a^2}{3}}}{\sqrt{4{R_o}^2 - \frac{4a^2}{3} + \frac{a^2}{3}}}\right) = 2\pi - 6\sin^{-1}\left(\frac{\sqrt{3{R_o}^2 - a^2}}{\sqrt{4{R_o}^2 - a^2}}\right) = 2\,\pi - 6\sin^{-1}\left(\sqrt{\frac{3{R_o}^2 - a^2}{4{R_o}^2 - a^2}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3\left(\frac{R_o}{a}\right)^2 - 1}{4\left(\frac{R_o}{a}\right)^2 - 1}}\right) = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right)$$

$$\omega_T = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) \quad \dots\dots\dots\dots\dots (4)$$

Where, $x = R_o/a$ is some unknown ratio.

### 2.5. Radius of circumscribed sphere (Circum-radius)

Since, a regular n-gonal right antiprism is a closed surface consisting of 2 regular n-sided polygonal faces and 2n equilateral triangular faces therefore the sum of solid angles subtended by all the faces at the centre of polygonal antiprism must be equal to $4\pi$ sr according to HCR's Theory of Polygon [1]. Thus, the total solid angle subtended by all the faces at the centre O of the polygonal antiprism is given as follows

$$2(\text{Solid angle subtended by polygonal face}) + 2n(\text{Solid angle subtended by triangular face}) = 4\pi$$

$$2(\omega_n) + 2n(\omega_T) = 4\pi$$

$$2\left(2\pi - 2n\sin^{-1}\left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\right)\right) + 2n\left(2\pi - 6\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right)\right) = 4\pi$$

$$4\pi - 4n\sin^{-1}\left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\right) + 4n\pi - 12n\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) = 4\pi$$

$$\sin^{-1}\left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\right) + 3\sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) = \pi$$

$$\sin^{-1}\left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\right) + \sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) = 2\left(\frac{\pi}{2} - \sin^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right)\right)$$

$$\cos^{-1}\left(\sqrt{1 - \left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\right)^2}\sqrt{1 - \left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right)^2} - \sqrt{\frac{4x^2\sin^2\frac{\pi}{n} - 1}{4x^2 - 1}}\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right) = 2\cos^{-1}\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right)$$

$$\cos^{-1}\left(\sqrt{\frac{4x^2 - 1 - 4x^2\sin^2\frac{\pi}{n} + 1}{4x^2 - 1}}\sqrt{\frac{4x^2 - 1 - 3x^2 + 1}{4x^2 - 1}} - \frac{\sqrt{\left(4x^2\sin^2\frac{\pi}{n} - 1\right)(3x^2 - 1)}}{4x^2 - 1}\right) = \cos^{-1}\left(2\left(\sqrt{\frac{3x^2 - 1}{4x^2 - 1}}\right)^2 - 1\right)$$

$$\cos^{-1}\left(\sqrt{\frac{4x^2\cos^2\frac{\pi}{n}}{4x^2 - 1}}\sqrt{\frac{x^2}{4x^2 - 1}} - \frac{\sqrt{\left(4x^2\sin^2\frac{\pi}{n} - 1\right)(3x^2 - 1)}}{4x^2 - 1}\right) = \cos^{-1}\left(\frac{6x^2 - 2 - 4x^2 + 1}{4x^2 - 1}\right)$$

$$\cos^{-1}\left(\frac{2x^2\cos\frac{\pi}{n}}{4x^2 - 1} - \frac{\sqrt{\left(4x^2\sin^2\frac{\pi}{n} - 1\right)(3x^2 - 1)}}{4x^2 - 1}\right) = \cos^{-1}\left(\frac{2x^2 - 1}{4x^2 - 1}\right)$$

$$\cos^{-1}\left(\frac{2x^2\cos\frac{\pi}{n} - \sqrt{\left(4x^2\sin^2\frac{\pi}{n} - 1\right)(3x^2 - 1)}}{4x^2 - 1}\right) = \cos^{-1}\left(\frac{2x^2 - 1}{4x^2 - 1}\right)$$

$$\frac{2x^2\cos\frac{\pi}{n} - \sqrt{\left(4x^2\sin^2\frac{\pi}{n} - 1\right)(3x^2 - 1)}}{4x^2 - 1} = \frac{2x^2 - 1}{4x^2 - 1}$$

$$2x^2\cos\frac{\pi}{n}-\sqrt{\left(4x^2\sin^2\frac{\pi}{n}-1\right)(3x^2-1)}-2x^2+1=0 \qquad \left(\forall\ |x|\neq\frac{1}{2}\right)$$

$$1-2x^2\left(1-\cos\frac{\pi}{n}\right)=\sqrt{\left(4x^2\sin^2\frac{\pi}{n}-1\right)(3x^2-1)}$$

$$\left(1-2x^2\left(1-\cos\frac{\pi}{n}\right)\right)^2=\left(\sqrt{\left(4x^2\sin^2\frac{\pi}{n}-1\right)(3x^2-1)}\right)^2 \qquad (\because \text{LHS}, \text{RHS}>0)$$

$$1+4x^4\ \left(1-\cos\frac{\pi}{n}\right)^2-4x^2\left(1-\cos\frac{\pi}{n}\right)=\left(4x^2\sin^2\frac{\pi}{n}-1\right)(3x^2-1)$$

$$1+4x^4\ \left(1-\cos\frac{\pi}{n}\right)^2-4x^2\left(1-\cos\frac{\pi}{n}\right)=12x^4\sin^2\frac{\pi}{n}-3x^2-4x^2\sin^2\frac{\pi}{n}+1$$

$$12x^4\sin^2\frac{\pi}{n}-4x^4\ \left(1-\cos\frac{\pi}{n}\right)^2+4x^2\left(1-\cos\frac{\pi}{n}\right)-4x^2\sin^2\frac{\pi}{n}-3x^2=0$$

$$x^2\left[4x^2\left(3\sin^2\frac{\pi}{n}-\left(1-\cos\frac{\pi}{n}\right)^2\right)-4\sin^2\frac{\pi}{n}-4\cos\frac{\pi}{n}+1\right]=0$$

$$4x^2\left(3\sin^2\frac{\pi}{n}-\left(1-\cos\frac{\pi}{n}\right)^2\right)-4\sin^2\frac{\pi}{n}-4\cos\frac{\pi}{n}+1=0 \qquad (\forall\ x\neq 0)$$

$$4x^2\left(3-3\cos^2\frac{\pi}{n}-1-\cos^2\frac{\pi}{n}+2\cos\frac{\pi}{n}\right)=4-4\cos^2\frac{\pi}{n}+4\cos\frac{\pi}{n}-1$$

$$8x^2\left(1+\cos\frac{\pi}{n}-2\cos^2\frac{\pi}{n}\right)=3+4\cos\frac{\pi}{n}-4\cos^2\frac{\pi}{n}$$

$$x^2=\frac{3+4\cos\frac{\pi}{n}-4\cos^2\frac{\pi}{n}}{8\left(1+\cos\frac{\pi}{n}-2\cos^2\frac{\pi}{n}\right)}=\frac{\left(1+2\cos\frac{\pi}{n}\right)\left(3-2\cos\frac{\pi}{n}\right)}{8\left(1+2\cos\frac{\pi}{n}\right)\left(1-\cos\frac{\pi}{n}\right)}=\frac{3-2\cos\frac{\pi}{n}}{8\left(1-\cos\frac{\pi}{n}\right)}$$

$$=\frac{3-2\left(1-2\sin^2\frac{\pi}{2n}\right)}{8\left(1-1+2\sin^2\frac{\pi}{2n}\right)}=\frac{1+4\sin^2\frac{\pi}{2n}}{16\sin^2\frac{\pi}{2n}}=\frac{1}{16}\left(4+\operatorname{cosec}^2\frac{\pi}{2n}\right)$$

$$x=\frac{1}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{2n}} \qquad (\because\ x\neq 0)$$

$$\Rightarrow\ \frac{R_o}{a}=\frac{1}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{2n}} \qquad \left(\because\ x=\frac{R_o}{a}\right)$$

$$\therefore\ \boldsymbol{R_o=\frac{a}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{2n}}} \qquad (\forall\, n\geq 3,\ \ n\in N) \qquad \ldots\ldots\ldots\ldots(5)$$

The above Eq(5) is the generalized formula to analytically compute the radius of circuminscribed sphere on which all 2n identical vertices of a regular n-gonal right antiprism with edge length $a$ lie.

Now, substituting the value of $R_o$ into Eq(1) above, the normal distance $H_n$ of regular n-sided polygonal face from the centre O of the polygonal antiprism is obtained as follows

$$H_n=\sqrt{{R_o}^2-\frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}}=\sqrt{\left(\frac{a}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{2n}}\right)^2-\frac{a^2}{4}\operatorname{cosec}^2\frac{\pi}{n}}=\frac{a}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{2n}-4\operatorname{cosec}^2\frac{\pi}{n}}$$

$$= \frac{a}{4}\sqrt{4+\frac{1}{\sin^2\frac{\pi}{2n}}-\frac{4}{\sin^2\frac{\pi}{n}}} = \frac{a}{4}\sqrt{4+\frac{1}{\sin^2\frac{\pi}{2n}}-\frac{4}{4\sin^2\frac{\pi}{2n}\cos^2\frac{\pi}{2n}}} = \frac{a}{4}\sqrt{4+\frac{1}{\sin^2\frac{\pi}{2n}}-\frac{1}{\sin^2\frac{\pi}{2n}\cos^2\frac{\pi}{2n}}}$$

$$= \frac{a}{4}\sqrt{\frac{4\sin^2\frac{\pi}{2n}\cos^2\frac{\pi}{2n}+\cos^2\frac{\pi}{2n}-1}{\sin^2\frac{\pi}{2n}\cos^2\frac{\pi}{2n}}} = \frac{a}{4}\sqrt{\frac{4\sin^2\frac{\pi}{2n}\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}\cos^2\frac{\pi}{2n}}} = \frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}}$$

$$\therefore\ \boldsymbol{H_n = \frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}}} \qquad (\forall\, n \geq 3,\ n \in N) \qquad \ldots\ldots\ldots\ldots..(6)$$

The above Eq(6) is the generalized formula to analytically compute the normal distance of regular polygonal face from the centre of a regular n-gonal right antiprism having edge length $a$.

Now, substituting the value of $R_o$ into Eq(3) above, the normal distance $H_T$ of equilateral triangular face from the centre O of the polygonal antiprism is obtained as follows

$$H_T = \sqrt{{R_o}^2-\frac{a^2}{3}} = \sqrt{\left(\frac{a}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{2n}}\right)^2-\frac{a^2}{3}} = a\sqrt{\frac{1}{48}\left(12+3\operatorname{cosec}^2\frac{\pi}{2n}-16\right)}$$

$$= a\sqrt{\frac{1}{48}\left(3\operatorname{cosec}^2\frac{\pi}{2n}-4\right)} = \frac{a}{4\sqrt{3}}\sqrt{3\sec^2\frac{\pi}{2n}\cot^2\frac{\pi}{2n}-4} = \frac{a}{4\sqrt{3}}\cot\frac{\pi}{2n}\sqrt{3\sec^2\frac{\pi}{2n}-4\tan^2\frac{\pi}{2n}}$$

$$= \frac{a}{4\sqrt{3}}\cot\frac{\pi}{2n}\sqrt{3\sec^2\frac{\pi}{2n}-4\left(\sec^2\frac{\pi}{2n}-1\right)} = \frac{a}{4\sqrt{3}}\cot\frac{\pi}{2n}\sqrt{4-\sec^2\frac{\pi}{2n}}$$

$$\therefore\ \boldsymbol{H_T = \frac{a}{12}\sqrt{9\operatorname{cosec}^2\frac{\pi}{2n}-12} = \frac{a}{4\sqrt{3}}\cot\frac{\pi}{2n}\sqrt{4-\sec^2\frac{\pi}{2n}}} \qquad (\forall\, n \geq 3,\ n \in N) \qquad \ldots\ldots\ldots..(7)$$

The above Eq(7) is the generalized formula to analytically compute the normal distance of equilateral triangular face from the centre of a regular n-gonal right antiprism having edge length $a$.

### 2.6. Normal height of regular polygonal antiprism

The normal height $H$ of the regular n-gonal right antiprism i.e. perpendicular distance between its two regular n-polygonal faces is given as

$$H = 2(OO_1) = 2(H_n) = 2\left(\frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}}\right) = \frac{a}{2}\sqrt{4-\sec^2\frac{\pi}{2n}}$$

$$\Rightarrow\ \boldsymbol{H = \frac{a}{2}\sqrt{4-\sec^2\frac{\pi}{2n}}} \qquad (\forall\, n \geq 3,\ n \in N) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

### 2.7. Surface area of regular polygonal antiprism

The total surface of a regular n-gonal right antiprism consists of two identical regular polygonal faces and 2n identical equilateral triangular faces all with an equal side $a$. Therefore the total surface area of the regular polygonal antiprism with edge length $a$ is the sum of all its (2n+2) faces, which is given as follows

$$A_s = 2(\text{Area of regular polygonal face}) + 2n(\text{Area of regular triangular face})$$

$$= 2\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right) + 2n\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)\Big|_{n=3} \qquad \text{(Where, } n = \text{number of sides in a regular polygon)}$$

$$= \frac{1}{2}na^2\cot\frac{\pi}{n} + 2n\left(\frac{1}{4}\cdot 3a^2\cot\frac{\pi}{3}\right) = \frac{1}{2}na^2\cot\frac{\pi}{n} + \frac{\sqrt{3}}{2}na^2 = \frac{1}{2}na^2\left(\sqrt{3} + \cot\frac{\pi}{n}\right)$$

Therefore, the total surface area of regular n-gonal right antiprism having edge length $a$, is given as follows

$$\boldsymbol{A_s = \frac{1}{2}na^2\left(\sqrt{3} + \cot\frac{\pi}{n}\right)} \qquad \dots\dots\dots\dots\dots\dots\dots\dots \ (9)$$

### 2.8. Volume of regular polygonal antiprism

The regular n-gonal right antiprism, having 2n+2 faces, is a convex polyhedron therefore it can be divided into 2n+2 number of elementary right pyramids out of which 2n have regular triangular base and two have regular polygonal base. The sum of volumes of all 2n+2 elementary right pyramids is equal to the volume of polygonal antiprism. Now, consider a regular polygonal antiprism with edge length $a$. The side of base of all elementary right pyramids is $a$ and the vertical heights of regular triangular and polygonal right pyramids are $H_T$ and $H_n$ respectively (as shown in the Figure-3). The volume of regular polygonal antiprism is given as follows

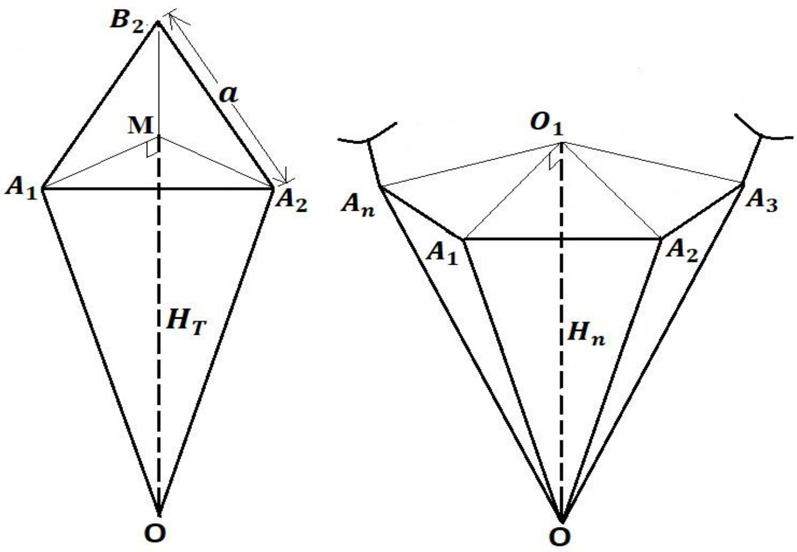


**Figure-3: A regular n-gonal right antiprism has 2n identical elementary right pyramids each with regular triangular base (left) and 2 identical elementary right pyramids each with regular n-gonal base (right).**

$$V = 2n(\text{Volume of regular triangular pyramid}) + 2(\text{Volume of regular polygonal pyramid})$$

$$= 2n\left(\frac{1}{3}\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)_{n=3}(H_T)\right) + 2\left(\frac{1}{3}\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)(H_n)\right) \qquad \text{(where, } n = \text{no. of sides in polygon)}$$

$$= 2n\left(\frac{1}{3}\left(\frac{1}{4}\cdot 3a^2\cot\frac{\pi}{3}\right)\cdot\frac{a}{4\sqrt{3}}\cot\frac{\pi}{2n}\sqrt{4-\sec^2\frac{\pi}{2n}}\right) + 2\left(\frac{1}{3}\left(\frac{1}{4}na^2\cot\frac{\pi}{n}\right)\cdot\frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}}\right) \quad \text{(setting values)}$$

$$= \frac{na^3}{24}\cot\frac{\pi}{2n}\sqrt{4-\sec^2\frac{\pi}{2n}} + \frac{na^3}{24}\cot\frac{\pi}{n}\sqrt{4-\sec^2\frac{\pi}{2n}}$$

$$= \frac{na^3}{24}\left(\cot\frac{\pi}{2n} + \cot\frac{\pi}{n}\right)\sqrt{4-\sec^2\frac{\pi}{2n}}$$

Therefore, the volume of regular n-gonal right antiprism having edge length $a$, is given as follows

$$\boldsymbol{V=\frac{na^3}{24}\left(\cot\frac{\pi}{2n}+\cot\frac{\pi}{n}\right)\sqrt{4-\sec^2\frac{\pi}{2n}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (10)$$

**2.9. Solid angle subtended by equilateral triangular face at the centre**

Now, substituting the value of $x=R_o/a$ from Eq(5) into Eq(4) above, the solid angle $\omega_T$ subtended by each regular triangular face at the centre of antiprism is obtained as follows

$$\omega_T=2\pi-6\sin^{-1}\left(\sqrt{\frac{3x^2-1}{4x^2-1}}\right)=2\pi-6\sin^{-1}\left(\sqrt{\frac{3\left(\frac{1}{4}\sqrt{4+\mathrm{cosec}^2\frac{\pi}{2n}}\right)^2-1}{4\left(\frac{1}{4}\sqrt{4+\mathrm{cosec}^2\frac{\pi}{2n}}\right)^2-1}}\right)$$

$$=2\pi-6\sin^{-1}\left(\sqrt{\frac{12+3\mathrm{cosec}^2\frac{\pi}{2n}-16}{16+4\mathrm{cosec}^2\frac{\pi}{2n}-16}}\right)=2\pi-6\sin^{-1}\left(\sqrt{\frac{3\mathrm{cosec}^2\frac{\pi}{2n}-4}{4\mathrm{cosec}^2\frac{\pi}{2n}}}\right)=2\pi-6\sin^{-1}\left(\sqrt{\frac{3}{4}-\sin^2\frac{\pi}{2n}}\right)$$

Therefore, the solid angle subtended by each regular triangular face at the centre of antiprism is given as

$$\boldsymbol{\omega_T=2\pi-6\sin^{-1}\left(\sqrt{\frac{3}{4}-\sin^2\frac{\pi}{2n}}\right)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (11)$$

**2.10. Solid angle subtended by regular polygonal face at the centre**

Now, substituting the value of $x=R_o/a$ from Eq(5) into Eq(2) above, the solid angle $\omega_n$ subtended by each regular polygonal face at the centre of antiprism is obtained as follows

$$\omega_n=2\pi-2n\sin^{-1}\left(\sqrt{\frac{4x^2\sin^2\frac{\pi}{n}-1}{4x^2-1}}\right)=2\pi-2n\sin^{-1}\left(\sqrt{\frac{4\left(\frac{1}{4}\sqrt{4+\mathrm{cosec}^2\frac{\pi}{2n}}\right)^2\sin^2\frac{\pi}{n}-1}{4\left(\frac{1}{4}\sqrt{4+\mathrm{cosec}^2\frac{\pi}{2n}}\right)^2-1}}\right)$$

$$=2\pi-2n\sin^{-1}\left(\sqrt{\frac{\left(16+4\mathrm{cosec}^2\frac{\pi}{2n}\right)\sin^2\frac{\pi}{n}-16}{16+4\mathrm{cosec}^2\frac{\pi}{2n}-16}}\right)$$

$$=2\pi-2n\sin^{-1}\left(\sqrt{\left(4\sin^2\frac{\pi}{2n}+1\right)4\sin^2\frac{\pi}{2n}\mathrm{cosec}^2\frac{\pi}{2n}-4\sin^2\frac{\pi}{2n}}\right)$$

$$=2\pi-2n\sin^{-1}\left(2\sin\frac{\pi}{2n}\sqrt{\left(4\sin^2\frac{\pi}{2n}+1\right)\left(1-\sin^2\frac{\pi}{2n}\right)-1}\right)$$

$$=2\pi-2n\sin^{-1}\left(2\sin\frac{\pi}{2n}\sqrt{3\sin^2\frac{\pi}{2n}-4\sin^4\frac{\pi}{2n}}\right)=2\pi-2n\sin^{-1}\left(2\sin^2\frac{\pi}{2n}\sqrt{3-4\sin^2\frac{\pi}{2n}}\right)$$

Therefore, the solid angle subtended by each regular polygonal face at the centre of antiprism is given as

$$\boldsymbol{\omega_n = 2\pi - 2n\sin^{-1}\left(2\sin^2\frac{\pi}{2n}\sqrt{3 - 4\sin^2\frac{\pi}{2n}}\right)} \quad \dots\dots\dots\dots\dots\dots\dots\dots \ (12)$$

**2.11. Solid angle subtended by the antiprism at one of its 2n identical vertices**

The vertices $A_2$ and $A_n$ are joined to the centre $O_1$ of regular polygonal face $A_1A_2A_3 \dots\dots A_{n-1}A_n$ to obtain isosceles $\Delta O_1A_nA_2$ having vertex $\angle A_nO_1A_2 = \frac{4\pi}{n}$. Now, in right $\Delta O_1QA_n$ (as shown in the Figure-4(a))

$$\sin\angle A_nO_1Q = \frac{A_nQ}{O_1A_n} \quad \Rightarrow \quad \sin\frac{2\pi}{n} = \frac{A_nQ}{\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}} \quad \Rightarrow \quad A_nQ = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}\sin\frac{2\pi}{n}$$

$$\boldsymbol{A_nQ} = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \boldsymbol{a\cos\frac{\pi}{n}} \quad \dots\dots\dots\dots\dots \ (13)$$

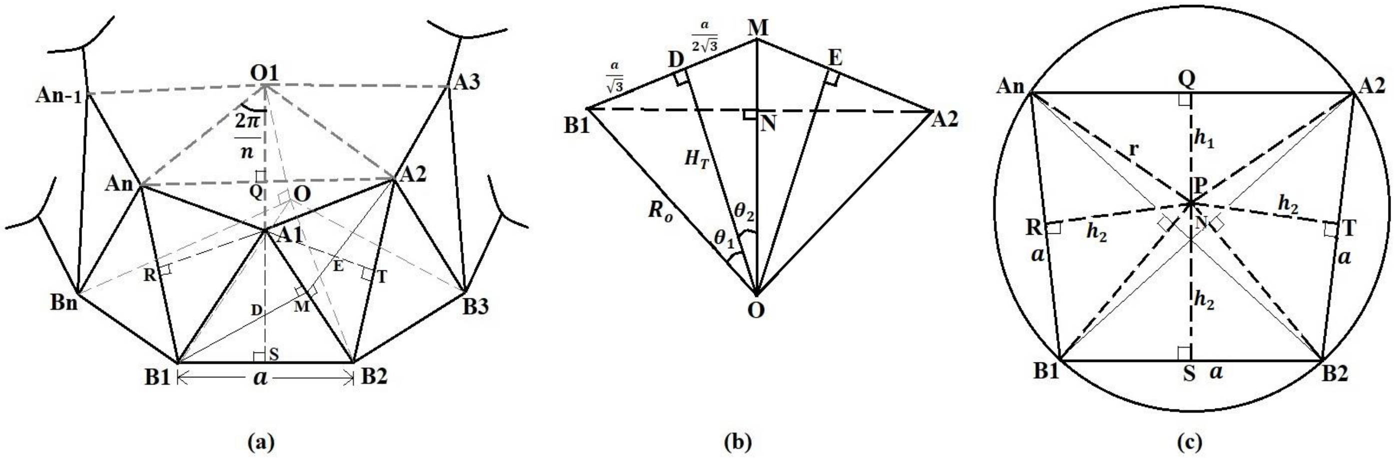


**Figure-4: (a) A regular n-gonal right antiprism (b) perpendiculars OD & OE are drawn from the center O to the adjacent equilateral triangular faces $A_1B_1B_2$ and $A_1A_2B_2$ having a common edge $A_1B_2$ (c) cyclic quadrilateral $B_1B_2\ A_2A_n$ is obtained by joining four vertices of the antiprism.**

The perpendiculars OD and OE are dropped from the centre O of the antiprism to the adjacent equilateral triangular faces $A_1B_1B_2$ and $A_1A_2B_2$, having a common edge $A_1B_2$, to their circum-centers/in-centers D and E respectively (as shown in the Figure-4(b)).

Now, in right $\Delta ODB_1$ (see above Fig-4(b)),

$$\tan\theta_1 = \frac{B_1D}{OD} = \frac{\frac{a}{\sqrt{3}}}{H_T} = \frac{\frac{a}{\sqrt{3}}}{\frac{a}{12}\sqrt{9\operatorname{cosec}^2\frac{\pi}{2n} - 12}} = \frac{4}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}$$

Similarly, in right $\Delta ODM$ (see above Fig-4(b)),

$$\tan\theta_2 = \frac{DM}{OD} = \frac{\frac{a}{2\sqrt{3}}}{H_T} = \frac{\frac{a}{2\sqrt{3}}}{\frac{a}{12}\sqrt{9\operatorname{cosec}^2\frac{\pi}{2n} - 12}} = \frac{2}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}$$

In right $\Delta ONB_1$ (see above Fig-4(b)),

$$\sin\angle B_1\ \mathrm{ON} = \frac{B_1N}{OB_1} \quad \Rightarrow \quad B_1N = OB_1 \sin\angle B_1\ \mathrm{ON} = R_o \sin(\theta_1 + \theta_2)$$

$$\boldsymbol{B_1N} = \frac{a}{4}\sqrt{4 + \operatorname{cosec}^2\frac{\pi}{2n}}\,(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

$$= \frac{a}{4}\sqrt{4 + \operatorname{cosec}^2\frac{\pi}{2n}}\left(\frac{\tan\theta_1}{\sqrt{1+\tan^2\theta_1}}\frac{1}{\sqrt{1+\tan^2\theta_2}} + \frac{1}{\sqrt{1+\tan^2\theta_1}}\frac{\tan\theta_2}{\sqrt{1+\tan^2\theta_2}}\right)$$

$$= \frac{a}{4}\sqrt{4 + \operatorname{cosec}^2\frac{\pi}{2n}}\left(\frac{\frac{4}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}}{\sqrt{1+\left(\frac{4}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}\right)^2}}\frac{1}{\sqrt{1+\left(\frac{2}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}\right)^2}} + \frac{1}{\sqrt{1+\left(\frac{4}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}\right)^2}}\frac{\frac{2}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}}{\sqrt{1+\left(\frac{2}{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}\right)^2}}\right)$$

$$= \frac{a}{4}\sqrt{4 + \operatorname{cosec}^2\frac{\pi}{2n}}\left(\frac{4}{\sqrt{3}\sqrt{4 + \operatorname{cosec}^2\frac{\pi}{2n}}}\frac{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}{\sqrt{3}\operatorname{cosec}\frac{\pi}{2n}} + \frac{\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}}{\sqrt{3}\sqrt{4 + \operatorname{cosec}^2\frac{\pi}{2n}}}\frac{2}{\sqrt{3}\operatorname{cosec}\frac{\pi}{2n}}\right)$$

$$= \frac{a}{4}\left(\frac{4}{3}\sin\frac{\pi}{2n}\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4} + \frac{2}{3}\sin\frac{\pi}{2n}\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}\right)$$

$$= \frac{a}{2}\sin\frac{\pi}{2n}\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4} = \boldsymbol{\frac{a}{2}\sqrt{3 - 4\sin^2\frac{\pi}{2n}}}$$

A regular n-gonal right antiprism has 2n identical vertices say $A_1, A_2, A_3, \ldots\ldots, A_{n-1}, A_n$ & $B_1, B_2, B_3, \ldots\ldots, B_{n-1}, B_n$ all lying on a sphere (i.e. circumscribed sphere with radius $R_o$). A cyclic quadrilateral $B_1B_2A_2A_n$ (see Figure-4(c) above) is obtained by joining the vertices $B_1, B_2, A_2,$ & $A_n$ (Figure-4(a)). The foot P of perpendicular $A_1P$ drawn from the vertex $A_1$ to the quadrilateral $B_1B_2A_2A_n$ will be at an equal distance $r$ (i.e. circum-radius) from the vertices $B_1, B_2, A_2, \& A_n$ . The perpendiculars PQ, PR, PS, and PT are drawn from the foot of perpendicular (F.O.P.) P to all the sides $A_2A_n$, $A_nB_1$, $B_1B_2$ and $B_2A_2$ respectively (as shown in the above Figure-4(c)).

From A-A similarity, right triangles $\Delta B_1NB_2$ and $\Delta PSB_2$ are similar triangles (see the above Fig-4(c) or Fig-5). Therefore, using the ratio of corresponding sides of the similar triangles

$$\frac{PS}{SB_2} = \frac{B_1N}{NB_2}$$

$$\frac{h_2}{a/2} = \frac{\frac{a}{2}\sqrt{3 - 4\sin^2\frac{\pi}{2n}}}{\frac{a}{2}\sqrt{1 + 4\sin^2\frac{\pi}{2n}}}$$

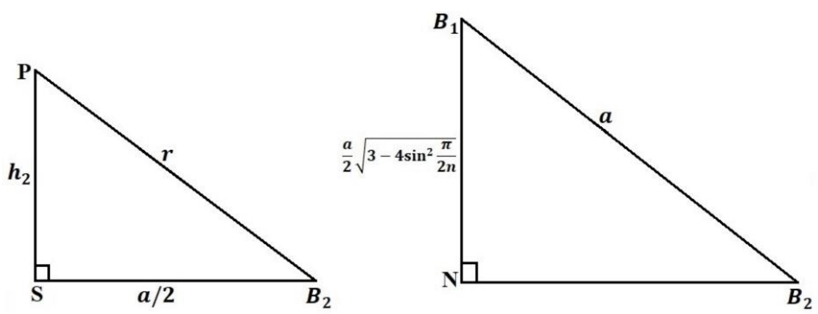


**Figure-5: $\Delta B_1NB_2$ and $\Delta PSB_2$ are similar triangles from A-A similarity.**

$$\boldsymbol{h_2} = \boldsymbol{\frac{a}{2}\sqrt{\frac{3 - 4\sin^2\frac{\pi}{2n}}{1 + 4\sin^2\frac{\pi}{2n}}}} \qquad \ldots (14)$$

In right $\Delta PSB_2$ (see Fig-5), using Pythagoras theorem as follows

$$r = PB_2 = \sqrt{(PS)^2 + (SB_2)^2} = \sqrt{(h_2)^2 + (a/2)^2} = \sqrt{\left(\frac{a}{2}\sqrt{\frac{3 - 4\sin^2\frac{\pi}{2n}}{1 + 4\sin^2\frac{\pi}{2n}}}\right)^2 + \frac{a^2}{4}}$$

$$\boldsymbol{r} = \frac{a}{2}\sqrt{\frac{3 - 4\sin^2\frac{\pi}{2n}}{1 + 4\sin^2\frac{\pi}{2n}} + 1} = \frac{a}{2}\sqrt{\frac{3 - 4\sin^2\frac{\pi}{2n} + 1 + 4\sin^2\frac{\pi}{2n}}{1 + 4\sin^2\frac{\pi}{2n}}} = \frac{\boldsymbol{a}}{\sqrt{\mathbf{1 + 4\sin^2}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}}$$

In right $\Delta PQA_n$ (see Fig-4(c) above), using Pythagoras theorem as follows

$$\boldsymbol{h_1} = PQ = \sqrt{(PA_n)^2 - (A_nQ)^2} = \sqrt{(r)^2 - \left(\frac{A_nA_2}{2}\right)^2} = \sqrt{\left(\frac{a}{\sqrt{1 + 4\sin^2\frac{\pi}{2n}}}\right)^2 - \left(a\cos\frac{\pi}{n}\right)^2}$$

$$= a\sqrt{\frac{1}{1 + 4\sin^2\frac{\pi}{2n}} - \cos^2\frac{\pi}{n}} = a\sqrt{\frac{1 - \cos^2\frac{\pi}{n}\left(1 + 4\sin^2\frac{\pi}{2n}\right)}{1 + 4\sin^2\frac{\pi}{2n}}} = a\sqrt{\frac{1 - \left(1 - 2\sin^2\frac{\pi}{2n}\right)^2\left(1 + 4\sin^2\frac{\pi}{2n}\right)}{1 + 4\sin^2\frac{\pi}{2n}}}$$

$$= a\sqrt{\frac{12\sin^4\frac{\pi}{2n} - 16\sin^6\frac{\pi}{2n}}{1 + 4\sin^2\frac{\pi}{2n}}} = \boldsymbol{2a\sin^2\frac{\pi}{2n}}\sqrt{\frac{\mathbf{3 - 4\sin^2}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}{\mathbf{1 + 4\sin^2}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}} \quad \dots\dots\dots\dots(15)$$

A perpendicular $A_1P$ is dropped from the vertex $A_1$of antiprism to the circum-centre P of cyclic quadrilateral $B_1B_2A_2A_n$ (see Figure-4(c) above) to obtain a right $\Delta A_1PB_1$ (as shown in the Figure-6).

In right $\Delta A_1PB_1$ (see Fig-6), using Pythagoras theorem as follows

$$\boldsymbol{h} = A_1P = \sqrt{(A_1B_1)^2 - (B_1P)^2} = \sqrt{a^2 - r^2}$$

$$= \sqrt{a^2 - \left(\frac{a}{\sqrt{1 + 4\sin^2\frac{\pi}{2n}}}\right)^2} = a\sqrt{1 - \frac{1}{1 + 4\sin^2\frac{\pi}{2n}}}$$

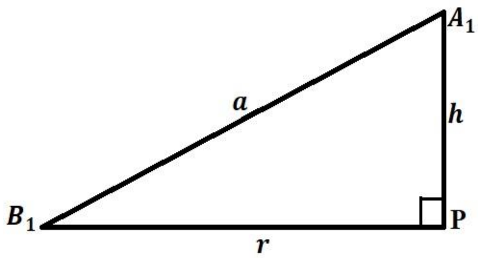


**Figure-6:** **Right $\Delta A_1PB_1$. The vertex $A_1$ is at a normal height $h$ above circum-centre P.**

$$= a\sqrt{\frac{4\sin^2\frac{\pi}{2n}}{1 + 4\sin^2\frac{\pi}{2n}}} = \frac{\boldsymbol{2a\sin\frac{\pi}{2n}}}{\sqrt{\mathbf{1 + 4\sin^2}\frac{\boldsymbol{\pi}}{\boldsymbol{2n}}}} \quad \dots\dots\dots\dots(16)$$

The solid angle subtended by a right triangle having legs $p$ and $b$ at any point lying on the axis passing through the acute angled vertex and perpendicular to the plane of triangle (as shown in the Figure-7 below), is given by standard formula of **HCR's Theory of Polygon** [3] as follows

$$\boldsymbol{\omega = \sin^{-1}\left(\frac{b}{\sqrt{b^2 + p^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2 + p^2}}\right)\left(\frac{h}{\sqrt{h^2 + p^2}}\right)\right\}} \quad \dots\dots\dots\dots\dots(17)$$

From the above Figure-4(c), the solid angle $\omega_{\Delta PQA_n}$ subtended by right $\Delta PQA_n$ , at the vertex $A_1$ of the antiprism is obtained by substituting the corresponding values, $b = A_nQ,\ p = h_1$ and $h = h$ from the above Eq(13), Eq(15) and Eq(16) respectively in the above standard formula (Eq(17)), is given as follows

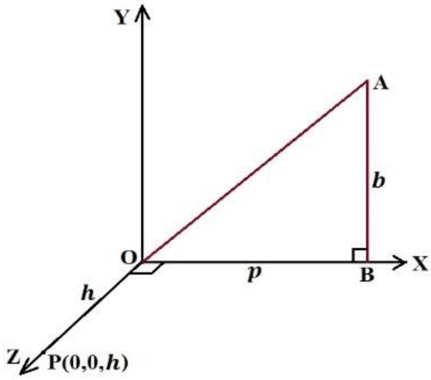


**Figure-7: The point $P(0,0,h)$ lies at a normal distance $h$ from acute angled vertex O of right $\Delta ABO$.**

$$\omega_{\Delta PQA_n} = \sin^{-1}\left(\frac{A_nQ}{\sqrt{A_nQ^2 + {h_1}^2}}\right) - \sin^{-1}\left\{\left(\frac{A_nQ}{\sqrt{A_nQ^2 + {h_1}^2}}\right)\left(\frac{h}{\sqrt{h^2 + {h_1}^2}}\right)\right\}$$

$$\omega_{\Delta PQA_n} = \sin^{-1}\left(\frac{a\cos\frac{\pi}{n}}{\sqrt{\left(a\cos\frac{\pi}{n}\right)^2 + \left(2a\sin^2\frac{\pi}{2n}\sqrt{\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}\right)^2}}\right)$$

$$-\sin^{-1}\left\{\left(\frac{a\cos\frac{\pi}{n}}{\sqrt{\left(a\cos\frac{\pi}{n}\right)^2 + \left(2a\sin^2\frac{\pi}{2n}\sqrt{\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}\right)^2}}\right)\left(\frac{\frac{2a\sin\frac{\pi}{2n}}{\sqrt{1+4\sin^2\frac{\pi}{2n}}}}{\sqrt{\left(\frac{2a\sin\frac{\pi}{2n}}{\sqrt{1+4\sin^2\frac{\pi}{2n}}}\right)^2 + \left(2a\sin^2\frac{\pi}{2n}\sqrt{\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{\cos\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{\sqrt{\left(1-2\sin^2\frac{\pi}{2n}\right)^2\left(1+4\sin^2\frac{\pi}{2n}\right) + 4\sin^4\frac{\pi}{2n}\left(3-4\sin^2\frac{\pi}{2n}\right)}}\right)$$

$$-\sin^{-1}\left\{\left(\frac{\cos\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{\sqrt{\left(1-2\sin^2\frac{\pi}{2n}\right)^2\left(1+4\sin^2\frac{\pi}{2n}\right) + 4\sin^4\frac{\pi}{2n}\left(3-4\sin^2\frac{\pi}{2n}\right)}}\right)\left(\frac{\frac{2a\sin\frac{\pi}{2n}}{\sqrt{1+4\sin^2\frac{\pi}{2n}}}}{\frac{2a\sin\frac{\pi}{2n}}{\sqrt{1+4\sin^2\frac{\pi}{2n}}}\sqrt{1+\sin^2\frac{\pi}{2n}\left(3-4\sin^2\frac{\pi}{2n}\right)}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{\cos\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{\sqrt{1+4\sin^2\frac{\pi}{2n}+4\sin^4\frac{\pi}{2n}+16\sin^6\frac{\pi}{2n}-4\sin^2\frac{\pi}{2n}-16\sin^4\frac{\pi}{2n}+12\sin^4\frac{\pi}{2n}-16\sin^6\frac{\pi}{2n}}}\right)$$

$$-\sin^{-1}\left\{\left(\frac{\cos\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{\sqrt{1+4\sin^2\frac{\pi}{2n}+4\sin^4\frac{\pi}{2n}+16\sin^6\frac{\pi}{2n}-4\sin^2\frac{\pi}{2n}-16\sin^4\frac{\pi}{2n}+12\sin^4\frac{\pi}{2n}-16\sin^6\frac{\pi}{2n}}}\right)\left(\frac{1}{\sqrt{1-\sin^2\frac{\pi}{2n}+4\sin^2\frac{\pi}{2n}-4\sin^4\frac{\pi}{2n}}}\right)\right\}$$

$$= \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}\right) - \sin^{-1}\left\{\left(\cos\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}\right)\left(\frac{1}{\sqrt{\left(1-\sin^2\frac{\pi}{2n}\right)\left(1+4\sin^2\frac{\pi}{2n}\right)}}\right)\right\}$$

$$= \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\sqrt{\cos^2\frac{\pi}{2n}}}\right) = \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right)$$

$$\omega_{\Delta PQA_n} = \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right) \quad \dots\dots\dots\dots (18)$$

Similarly, from the above Figure-4(c), the solid angle $\omega_{\Delta PRA_n}$ subtended by right $\Delta PRA_n$ , at the vertex $A_1$ of the antiprism is obtained by substituting the corresponding values, $b = a/2,\ p = h_2$ and $h = h$ from the above Eq(14) and Eq(16) in the above standard formula (Eq(17)), is given as follows

$$\omega_{\Delta PRA_n} = \sin^{-1}\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + {h_2}^2}}\right) - \sin^{-1}\left\{\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + {h_2}^2}}\right)\left(\frac{h}{\sqrt{h^2 + {h_2}^2}}\right)\right\}$$

$$\omega_{\Delta PRA_n} = \sin^{-1}\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\sqrt{\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{\frac{a}{2}}{\sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\sqrt{\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}\right)^2}}\right)\left(\frac{\frac{2a\sin\frac{\pi}{2n}}{\sqrt{1+4\sin^2\frac{\pi}{2n}}}}{\sqrt{\left(\frac{2a\sin\frac{\pi}{2n}}{\sqrt{1+4\sin^2\frac{\pi}{2n}}}\right)^2 + \left(\frac{a}{2}\sqrt{\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}\right)^2}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{\frac{a}{2}}{\frac{a}{2}\sqrt{1+\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}}\right) - \sin^{-1}\left\{\left(\frac{\frac{a}{2}}{\frac{a}{2}\sqrt{1+\frac{3-4\sin^2\frac{\pi}{2n}}{1+4\sin^2\frac{\pi}{2n}}}}\right)\left(\frac{\frac{2a\sin\frac{\pi}{2n}}{\sqrt{1+4\sin^2\frac{\pi}{2n}}}}{\frac{a}{2\sqrt{1+4\sin^2\frac{\pi}{2n}}}\sqrt{16\sin^2\frac{\pi}{2n}+3-4\sin^2\frac{\pi}{2n}}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{\frac{4}{1+4\sin^2\frac{\pi}{2n}}}}\right) - \sin^{-1}\left\{\left(\frac{1}{\sqrt{\frac{4}{1+4\sin^2\frac{\pi}{2n}}}}\right)\left(\frac{4\sin\frac{\pi}{2n}}{\sqrt{3}\sqrt{1+4\sin^2\frac{\pi}{2n}}}\right)\right\}$$

$$= \sin^{-1}\left(\frac{1}{2}\sqrt{1+4\sin^2\frac{\pi}{2n}}\right) - \sin^{-1}\left\{\left(\frac{1}{2}\sqrt{1+4\sin^2\frac{\pi}{2n}}\right)\left(\frac{4\sin\frac{\pi}{2n}}{\sqrt{3}\sqrt{1+4\sin^2\frac{\pi}{2n}}}\right)\right\}$$

$$\omega_{\Delta PRA_n} = \sin^{-1}\left(\frac{1}{2}\sqrt{1+4\sin^2\frac{\pi}{2n}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right) \quad \dots\dots\dots\dots\dots.. \ (19)$$

According to HCR's Theory of Polygon, the solid angle $\omega_{B_1B_2A_2A_n}$ subtended at the vertex $A_1$ of antiprism by the cyclic quadrilateral (trapezium) $B_1B_2A_2A_n$ will be equal to the algebraic sum of the solid angles subtended by the congruent right triangles $\Delta PQA_n$ & $\Delta PQA_2$ and the congruent right triangles $\Delta PRA_n$, $\Delta PRB_1$, $\Delta PSB_1$, $\Delta PSB_2$, $\Delta PTB_2$ & $\Delta PTA_2$ (as shown in the above Figure-4(c)). Therefore the solid angle subtended by the cyclic quadrilateral $B_1B_2A_2A_n$ at the vertex $A_1$ of polygonal antiprism is obtained as follows

$$\omega_{B_1B_2A_2A_n} = \omega_{\Delta PQA_n} + \omega_{\Delta PQA_2} + \omega_{\Delta PRA_n} + \omega_{\Delta PRB_1} + \omega_{\Delta PSB_1} + \omega_{\Delta PSB_2} + \omega_{\Delta PTB_2} + \omega_{\Delta PTA_2}$$

$$\omega_{B_1B_2A_2A_n} = 2\left(\omega_{\Delta PQA_n}\right) + 6\left(\omega_{\Delta PRA_n}\right) \quad (\because \text{ triangles are congruent })$$

Substituting the corresponding values from the above Eq(18) and Eq(19) as follows

$$\omega_{B_1B_2A_2A_n} = 2\left(\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right)\right) + 6\left(\sin^{-1}\left(\frac{1}{2}\sqrt{1+4\sin^2\frac{\pi}{2n}}\right) - \sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right)$$

$$\omega_{B_1B_2A_2A_n} = 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3-2\cos\frac{\pi}{n}}\right) + \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right) - 3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right)$$

$$\omega_{B_1B_2A_2A_n} = 2\cos^{-1}\left(\frac{2\sin^3\frac{\pi}{2n}\left(3-4\sin^2\frac{\pi}{2n}\right) + \cos^2\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{\cos\frac{\pi}{2n}}\right) + 6\cos^{-1}\left(\frac{3-4\sin^2\frac{\pi}{2n} + 2\sin\frac{\pi}{2n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{2\sqrt{3}}\right)$$

According to HCR's Theory of Polygon, the cone of vision of an object at a given point (i.e. eye of observer) in 3D space is an imaginary cone obtained by joining all the points of that object to the given point and the solid angle subtended by the object at that point is equal to the solid angle subtended by its cone of vision at the apex (i.e. eye of observer) [3]. It is worth noticing that cones of vision of regular polygonal right antiprism and the cyclic quadrilateral $B_1B_2A_2A_n$ are same therefore the solid angles subtended by the regular polygonal right antiprism and the cyclic quadrilateral $B_1B_2A_2A_n$ at the vertex $A_1$ will be equal (as shown in the above Figure-4).

Therefore, the solid angle subtended by regular polygonal right antiprism at its vertex is given as

$$\boldsymbol{\omega_{\mathrm{V}} = 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3-2\cos\frac{\pi}{n}}\right) + \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right) - 3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right)} \quad \ldots\ldots (20)$$

Or

$$\boldsymbol{\omega_{\mathrm{V}} = 2\cos^{-1}\left(\frac{2\sin^3\frac{\pi}{2n}\left(3-4\sin^2\frac{\pi}{2n}\right) + \cos^2\frac{\pi}{n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{\cos\frac{\pi}{2n}}\right) + 6\cos^{-1}\left(\frac{3-4\sin^2\frac{\pi}{2n} + 2\sin\frac{\pi}{2n}\sqrt{1+4\sin^2\frac{\pi}{2n}}}{2\sqrt{3}}\right)}$$

It is interesting to note that the above value of solid angle $\omega_{\mathrm{V}}$ is independent of the edge length $a$ of regular polygonal right antiprism but depends only on the number of sides $n$ of regular polygonal base.

### 2.12. Dihedral angle between any two adjacent equilateral triangular faces sharing a common edge

Let's consider any two adjacent equilateral triangular faces $A_1B_1B_2$ and $A_1A_2B_2$ having a common edge $A_1B_2$ (as shown in the top view of Figure-8 below). Drop the perpendiculars OD and OE from the centre O of the antiprism to the triangular faces $A_1B_1B_2$ and $A_1A_2B_2$, respectively which meet the faces at their in-centres D and E, respectively.

In right $\Delta ODM$ (see the front view in the Figure-8 below),

$$\tan\angle DMO = \frac{OD}{DM}$$

$$\Rightarrow \tan\frac{\theta_{TTE}}{2} = \frac{H_T}{DM} \qquad \left(\because \angle DMO = \frac{\angle DME}{2} = \frac{\theta_{TTE}}{2}\right)$$

$$\tan\frac{\theta_{TTE}}{2} = \frac{\frac{a}{12}\sqrt{9\operatorname{cosec}^2\frac{\pi}{2n} - 12}}{\frac{a}{2\sqrt{3}}} = \frac{1}{2}\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4} \qquad \left(\because DM = \frac{a}{2\sqrt{3}}\right)$$

$$\theta_{TTE} = 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n} - 4}\right)$$

Therefore the dihedral angle $\theta_{TTE}$ between any two adjacent regular triangular faces sharing a common edge in a regular pentagonal antiprism is given as

$$\boldsymbol{\theta_{TTE} = 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n} - 4}\right)} \quad \dots\dots\dots (21)$$

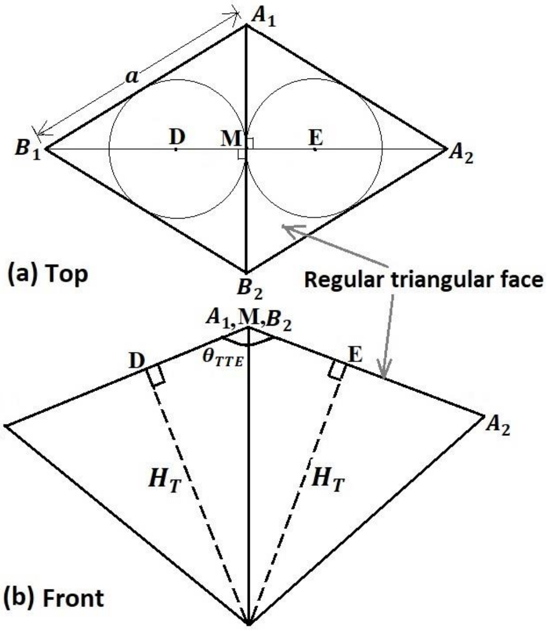


**Figure-8: (a) Two adjacent regular triangular faces having a common edge $A_1B_2$ (b) The perpendiculars drawn from the center O to the triangular faces fall at their in-centers D & E, and their in-circles touch each other at the mid-point M of common edge $A_1B_2$.**

### 2.13. Dihedral angle between regular triangular and polygonal faces sharing a common edge

Let's consider any two adjacent regular triangular and polygonal faces $A_1A_2B_2$ and $A_1A_2A_3 \dots . A_n$ having a common edge $A_1A_2$ (as shown in the top view in the Figure-9 below). Drop the perpendiculars OE and $OO_1$ from the centre O of the antiprism to the triangular face and polygonal face respectively which meet the faces at their in-centres E and $O_1$, respectively. The inscribed radius of equilateral triangular face $A_1A_2B_2$ is $\frac{a}{2\sqrt{3}}$.

In right $\Delta OEN$ (see the front view in the Figure-9),

$$\tan\angle ENO = \frac{EO}{EN} \quad \Rightarrow \tan\theta_1 = \frac{H_T}{\frac{a}{2\sqrt{3}}}$$

$$\tan\theta_1 = \frac{\frac{a}{12}\sqrt{9\mathrm{cosec}^2\frac{\pi}{2n} - 12}}{\frac{a}{2\sqrt{3}}}$$

$$\theta_1 = \tan^{-1}\left(\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n} - 4}\right)$$

Now, the inscribed radius $r$ of regular polygonal face $A_1A_2A_3 \dots . A_n$ with side $a$ is given by generalized formula as follows

$$r = \frac{a}{2}\cot\frac{\pi}{n}$$

In right $\Delta OO_1N$ (see the front view in the Figure-9)

$$\tan\angle O_1NO = \frac{OO_1}{NO_1} \quad \Rightarrow \tan\theta_2 = \frac{H_n}{r}$$

$$\tan\theta_2 = \frac{\frac{a}{4}\sqrt{4 - \sec^2\frac{\pi}{2n}}}{\frac{a}{2}\cot\frac{\pi}{n}} = \frac{1}{2}\tan\frac{\pi}{n}\sqrt{4 - \sec^2\frac{\pi}{2n}}$$

$$\theta_2 = \tan^{-1}\left(\frac{1}{2}\tan\frac{\pi}{n}\sqrt{4 - \sec^2\frac{\pi}{2n}}\right)$$

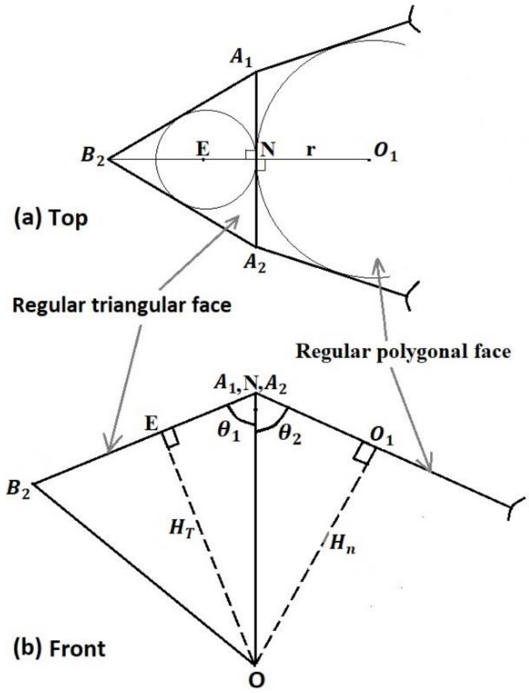


**Figure-9: (a) Two adjacent regular triangular and polygonal faces having a common edge $A_1A_2$ (b) The perpendiculars drawn from the center O to the faces fall at their in-centers E & $O_1$, and their in-circles touch each other at the mid-point N of common edge $A_1A_2$.**

Now, the total dihedral angle $\theta_{TPE}$ between regular triangular and polygonal faces with a common edge is the sum of dihedral angles $\theta_1$ and $\theta_2$ as determined above (see the front view in the above Figure-9) as follows

$$\theta_{TPE} = \theta_1 + \theta_2 = \tan^{-1}\left(\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n} - 4}\right) + \tan^{-1}\left(\frac{1}{2}\tan\frac{\pi}{n}\sqrt{4 - \sec^2\frac{\pi}{2n}}\right)$$

$$\theta_{TPE} = \cos^{-1}\left(\frac{1}{\sqrt{1+\left(\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n}-4}\right)^2}}\frac{1}{\sqrt{1+\left(\frac{1}{2}\tan\frac{\pi}{n}\sqrt{4-\sec^2\frac{\pi}{2n}}\right)^2}} - \frac{\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n}-4}}{\sqrt{1+\left(\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n}-4}\right)^2}}\frac{\frac{1}{2}\tan\frac{\pi}{n}\sqrt{4-\sec^2\frac{\pi}{2n}}}{\sqrt{1+\left(\frac{1}{2}\tan\frac{\pi}{n}\sqrt{4-\sec^2\frac{\pi}{2n}}\right)^2}}\right)$$

$$= \cos^{-1}\left(\frac{2}{\sqrt{4+3\mathrm{cosec}^2\frac{\pi}{2n}-4}}\frac{2}{\sqrt{4+\tan^2\frac{\pi}{n}\left(4-\sec^2\frac{\pi}{2n}\right)}} - \frac{\frac{\sqrt{3-4\sin^2\frac{\pi}{2n}}}{\sin\frac{\pi}{2n}}}{\sqrt{4+3\mathrm{cosec}^2\frac{\pi}{2n}-4}}\frac{\frac{\sin\frac{\pi}{n}\sqrt{4\cos^2\frac{\pi}{2n}-1}}{\cos\frac{\pi}{n}\cos\frac{\pi}{2n}}}{\sqrt{4+\tan^2\frac{\pi}{n}\left(4-\sec^2\frac{\pi}{2n}\right)}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n}}\sqrt{4+\frac{\sin^2\frac{\pi}{n}}{\cos^2\frac{\pi}{n}}\left(4-\frac{1}{\cos^2\frac{\pi}{2n}}\right)}} - \frac{\sin\frac{\pi}{n}\sqrt{3-4\sin^2\frac{\pi}{2n}}\sqrt{4\cos^2\frac{\pi}{2n}-1}}{\sin\frac{\pi}{2n}\cos\frac{\pi}{n}\cos\frac{\pi}{2n}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n}}\sqrt{4+\frac{\sin^2\frac{\pi}{n}}{\cos^2\frac{\pi}{n}}\left(4-\frac{1}{\cos^2\frac{\pi}{2n}}\right)}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{3}\,\mathrm{cosec}\frac{\pi}{2n}\sqrt{\frac{4\left(\cos^2\frac{\pi}{n}+\sin^2\frac{\pi}{n}\right)\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{n}}{\cos^2\frac{\pi}{n}\cos^2\frac{\pi}{2n}}}} - \frac{\sin\frac{\pi}{n}\sqrt{\left(3-4\sin^2\frac{\pi}{2n}\right)\left(4\cos^2\frac{\pi}{2n}-1\right)}}{\sin\frac{\pi}{2n}\cos\frac{\pi}{n}\cos\frac{\pi}{2n}\sqrt{3}\,\mathrm{cosec}\frac{\pi}{2n}\sqrt{\frac{4\left(\cos^2\frac{\pi}{n}+\sin^2\frac{\pi}{n}\right)\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{n}}{\cos^2\frac{\pi}{n}\cos^2\frac{\pi}{2n}}}}\right)$$

$$= \cos^{-1}\left(\frac{4}{\sqrt{3}\,\mathrm{cosec}\frac{\pi}{2n}\frac{\sqrt{4\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{n}}}{\cos\frac{\pi}{n}\cos\frac{\pi}{2n}}} - \frac{\sin\frac{\pi}{n}\sqrt{\left(3-4+4\cos^2\frac{\pi}{2n}\right)\left(4\cos^2\frac{\pi}{2n}-1\right)}}{\sqrt{3}\cos\frac{\pi}{n}\cos\frac{\pi}{2n}\frac{\sqrt{4\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{n}}}{\cos\frac{\pi}{n}\cos\frac{\pi}{2n}}}\right)$$

$$= \cos^{-1}\left(\frac{4\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}\cos\frac{\pi}{n}}{\sqrt{3}\sqrt{4\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{n}}} - \frac{\sin\frac{\pi}{n}\sqrt{\left(4\cos^2\frac{\pi}{2n}-1\right)\left(4\cos^2\frac{\pi}{2n}-1\right)}}{\sqrt{3}\sqrt{4\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{n}}}\right)$$

$$= \cos^{-1}\left(\frac{2\sin\frac{\pi}{n}\cos\frac{\pi}{n} - \sin\frac{\pi}{n}\sqrt{\left(4\cos^2\frac{\pi}{2n}-1\right)^2}}{\sqrt{3}\sqrt{4\cos^2\frac{\pi}{2n}-\sin^2\frac{\pi}{n}}}\right)$$

$$= \cos^{-1}\left(\frac{\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\left|4\cos^2\frac{\pi}{2n}-1\right|}{\sqrt{3}\sqrt{2\left(1+\cos\frac{\pi}{n}\right)-\left(1-\cos^2\frac{\pi}{n}\right)}}\right)$$

$$= \cos^{-1}\left(\frac{\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\left(4\cos^2\frac{\pi}{2n}-1\right)}{\sqrt{3}\sqrt{2+2\cos\frac{\pi}{n}-1+\cos^2\frac{\pi}{n}}}\right) \qquad (\because\ n \geq 3, n \in N)$$

$$= \cos^{-1}\left(\frac{\sin\frac{2\pi}{n} - \sin\frac{\pi}{n}\left(2\left(1+\cos\frac{\pi}{n}\right)-1\right)}{\sqrt{3}\sqrt{\cos^2\frac{\pi}{n}+2\cos\frac{\pi}{n}+1}}\right)$$

$$= \cos^{-1}\left(\frac{\sin\frac{2\pi}{n} - 2\sin\frac{\pi}{n} - 2\sin\frac{\pi}{n}\cos\frac{\pi}{n} + \sin\frac{\pi}{n}}{\sqrt{3}\sqrt{\left(1+\cos\frac{\pi}{n}\right)^2}}\right) = \cos^{-1}\left(\frac{\sin\frac{2\pi}{n} - \sin\frac{2\pi}{n} - \sin\frac{\pi}{n}}{\sqrt{3}\left|1+\cos\frac{\pi}{n}\right|}\right)$$

$$= \cos^{-1}\left(\frac{-\sin\frac{\pi}{n}}{\sqrt{3}\left(1+\cos\frac{\pi}{n}\right)}\right) = \cos^{-1}\left(\frac{-2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}{\sqrt{3}\left(1+2\cos^2\frac{\pi}{2n}-1\right)}\right) = \cos^{-1}\left(\frac{-2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}}{2\sqrt{3}\cos^2\frac{\pi}{2n}}\right)$$

$$= \cos^{-1}\left(\frac{-\sin\frac{\pi}{2n}}{\sqrt{3}\cos\frac{\pi}{2n}}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)$$

Therefore the dihedral angle $\theta_{TPE}$ between any two adjacent regular triangular and polygonal faces having a common edge in a regular polygonal right antiprism is given as

$$\boldsymbol{\theta_{TPE} = \pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)} \quad \dots\dots\dots\dots\dots\dots\ (22)$$

### 2.14. Dihedral angle between regular triangular and polygonal faces sharing a common vertex

It's worth noticing that the dihedral angle $\theta_{TPV}$ between any two adjacent regular triangular and polygonal faces having a common vertex in a regular polygonal antiprism is supplementary angle of $\theta_{TPE}$ which is given as

$$\boldsymbol{\theta_{TPV} = \pi - \theta_{TPE} = \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)} \quad \dots\dots\dots\dots\dots\dots\dots\ (23)$$

### 2.15. Dihedral angle between any two adjacent equilateral triangular faces sharing a common vertex

If $\alpha, \beta,$ and $\gamma$ are the face angles i.e. the angles between the consecutive lateral edges meeting at the vertex O in a tetrahedron OABC then its internal dihedral angles say $\theta_1, \theta_2$ and $\theta_3$ opposite to the face angles $\alpha, \beta$ and $\gamma$ respectively (as shown in the Figure-10) are given by HCR's inverse cosine formula [4] as follows

$$\theta_1 = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) \quad \dots\dots\dots\ (24)$$

$$\theta_2 = \cos^{-1}\left(\frac{\cos\beta - \cos\alpha\cos\gamma}{\sin\alpha\sin\gamma}\right) \quad \dots\dots\dots\ (25)$$

$$\theta_3 = \cos^{-1}\left(\frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}\right) \quad \dots\dots\dots\ (26)$$

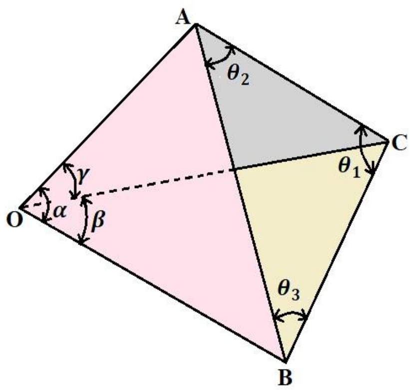


**Figure-10: Dihedral angles $\theta_1, \theta_2$ & $\theta_3$ are opposite to face angles $\alpha, \beta$ & $\gamma$ respectively in tetrahedron OABC**

Now, consider any two adjacent equilateral triangular faces say $A_1B_1A_n$ and $A_1B_2A_2$ having a common vertex $A_1$ and extend both the faces so that they intersect each other at the line segment $A_1B$ (as shown in the Figure-11 (a) below).

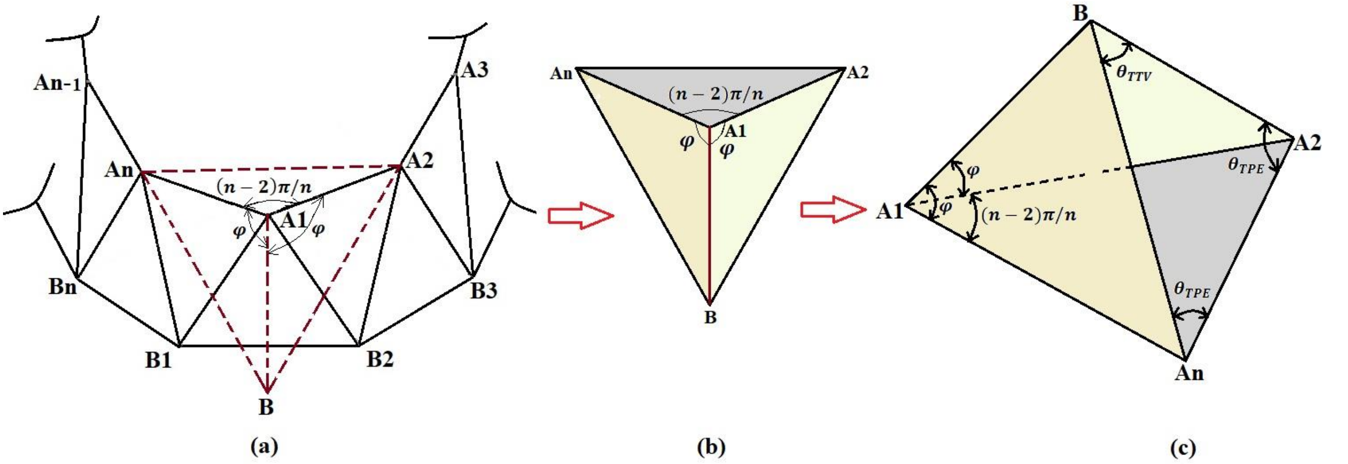


**Figure-11: (a) The triangular faces $A_1B_1A_n$ & $A_1B_2A_2$ are extended to make them intersect each other at the line segment $A_1B$ (b) tetrahedron $A_1A_2A_nB$ with face angles at the vertex $A_1$ (c) tetrahedron $A_1A_2A_nB$ with dihedral angles between its triangular faces meeting at their common edges**

Let $\theta_{TTV}$ be the dihedral angle between the equilateral triangular faces $A_1B_1A_n$ and $A_1B_2A_2$ about their common edge i.e. the line-segment of intersection $A_1B$ which is also the dihedral angle between the triangular faces $A_1BA_n$ and $A_1BA_2$ obtained by construction as shown in the above Figure-11(a).

It is also interesting to note that $\theta_{TPE}$ is the dihedral angle between any triangular face say $A_1B_2A_2$ and regular polygonal face $A_1A_2\ldots A_n$ which is also the dihedral angle between triangular faces $A_1A_2A_n$ & $A_1BA_2$, and dihedral angle between triangular faces $A_1A_2A_n$ & $A_1BA_2$ as shown in the above Figure-11(a). Now, assume that $\angle BA_1A_2 = \angle BA_1A_n = \varphi \ (0 < \varphi < \pi)$ using symmetry in the above Figure-11(a).

Now, consider the tetrahedron $A_1A_2A_nB$ with vertex $A_1$ at which three equilateral triangular faces $A_1A_2A_n$, $A_1A_nB$ and $A_1BA_2$ having vertex angles $(n-2)\pi/n,\ \varphi$ and $\varphi$ respectively (as shown in the above Figure-11(b)).

The angles $\theta_{TPE}$, $\theta_{TPE}$ and $\theta_{TTV}$ are the internal dihedral angles between triangular faces $A_1A_2A_n$ & $A_1A_nB$, $A_1A_2A_n$ & $A_1A_2B$, and $A_1A_nB$ & $A_1A_2B$ about the common edges $A_1A_n,\ A_1A_2$, and $A_1B$ respectively meeting at the vertex $A_1$. (as shown in the above Figure-11(c)).

Now, substituting $\alpha = \varphi, \beta = (n-2)\pi/n$ and $\gamma = \varphi$ in the above inverse cosine formula i.e. Eq(24), the internal dihedral angle $\theta_1 = \theta_{TPE}$ opposite to the face angle $\alpha = \varphi$ in tetrahedron $A_1A_2A_nB$ (see the above Figure-11(c)) is obtained as follows

$$\theta_1 = \cos^{-1}\left(\frac{\cos\alpha - \cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right) \quad \Rightarrow \ \theta_{TPE} = \cos^{-1}\left(\frac{\cos\varphi - \cos\frac{(n-2)\pi}{n}\cos\varphi}{\sin\frac{(n-2)\pi}{n}\sin\varphi}\right)$$

$$\frac{\cos\varphi - \cos\frac{(n-2)\pi}{n}\cos\varphi}{\sin\frac{(n-2)\pi}{n}\sin\varphi} = \cos\theta_{TPE}$$

$$\frac{\cos\varphi}{\sin\varphi}\left(\frac{1-\cos\frac{(n-2)\pi}{n}}{\sin\frac{(n-2)\pi}{n}}\right) = \cos\left(\pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\right) \qquad \text{(Setting value of } \theta_{TPE}\text{)}$$

$$\cot\varphi\left(\frac{1+\cos\frac{2\pi}{n}}{\sin\frac{2\pi}{n}}\right) = -\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\right)$$

$$\cot\varphi\left(\frac{1+2\cos^2\frac{\pi}{n}-1}{2\sin\frac{\pi}{n}\cos\frac{\pi}{n}}\right)=-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}$$

$$\frac{\cos\frac{\pi}{n}}{\sin\frac{\pi}{n}}=-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\tan\varphi$$

$$-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\tan\varphi=\cot\frac{\pi}{n}$$

$$\tan\varphi=-\sqrt{3}\ \cot\frac{\pi}{n}\cot\frac{\pi}{2n}$$

$$\Rightarrow\ \cos\varphi=-\frac{1}{\sqrt{1+\tan^2\varphi}}\qquad\left(\forall\ \frac{\pi}{2}<\varphi<\pi\right)$$

$$\cos\varphi=-\frac{1}{\sqrt{1+\left(-\sqrt{3}\ \cot\frac{\pi}{n}\cot\frac{\pi}{2n}\right)^2}}$$

$$\cos\varphi=-\frac{1}{\sqrt{1+3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n}}}\qquad\ldots\ldots\ldots(27)$$

Now, substituting $\alpha=(n-2)\pi/n$, $\beta=\varphi$ and $\gamma=\varphi$ in the above inverse cosine formula i.e. Eq(24), the internal dihedral angle $\theta_1=\theta_{TTV}$ opposite to the face angle $\alpha=(n-2)\pi/n$ in tetrahedron $A_1A_2A_nB$ (see the above Figure-11(c)) is obtained as follows

$$\theta_1=\cos^{-1}\left(\frac{\cos\alpha-\cos\beta\cos\gamma}{\sin\beta\sin\gamma}\right)$$

$$\Rightarrow\ \theta_{TTV}=\cos^{-1}\left(\frac{\cos\frac{(n-2)\pi}{n}-\cos\varphi\cos\varphi}{\sin\varphi\sin\varphi}\right)$$

$$=\cos^{-1}\left(\frac{\cos\frac{(n-2)\pi}{n}-\cos^2\varphi}{\sin^2\varphi}\right)$$

$$=\cos^{-1}\left(\frac{-\cos\frac{2\pi}{n}-\cos^2\varphi}{1-\cos^2\varphi}\right)$$

$$=\cos^{-1}\left(\frac{-\cos\frac{2\pi}{n}-\left(-\frac{1}{\sqrt{1+3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n}}}\right)^2}{1-\left(-\frac{1}{\sqrt{1+3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n}}}\right)^2}\right)$$

$$= \cos^{-1}\left(\frac{-\cos\frac{2\pi}{n} - \frac{1}{1+3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n}}}{1 - \frac{1}{1+3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n}}}\right) = \cos^{-1}\left(\frac{-\cos\frac{2\pi}{n}\left(1+3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n}\right) - 1}{1+3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n} - 1}\right)$$

$$= \cos^{-1}\left(\frac{-\cos\frac{2\pi}{n} - 3\cos\frac{2\pi}{n}\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n} - 1}{3\cot^2\frac{\pi}{n}\cot^2\frac{\pi}{2n}}\right) = \cos^{-1}\left(-\frac{1}{3}\left(1+\cos\frac{2\pi}{n}\right)\tan^2\frac{\pi}{n}\tan^2\frac{\pi}{2n} - \cos\frac{2\pi}{n}\right)$$

$$= \cos^{-1}\left(-\frac{1}{3}\left(2\cos^2\frac{\pi}{n}\right)\tan^2\frac{\pi}{n}\tan^2\frac{\pi}{2n} - \cos\frac{2\pi}{n}\right) = \cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{n}\tan^2\frac{\pi}{2n} - \cos\frac{2\pi}{n}\right)$$

Therefore the dihedral angle $\theta_{TTV}$ between any two adjacent regular triangular faces having a common vertex in a regular polygonal right antiprism is given as

$$\boldsymbol{\theta_{TTV} = \cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{n}\tan^2\frac{\pi}{2n} - \cos\frac{2\pi}{n}\right)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (28)$$

$\boldsymbol{\forall\ n \geq 3, \quad n \in N}$

It's worth noticing that the dihedral angles given by above Eq(21), Eq(22), Eq(23) and Eq(28) and the solid angles given by above Eq(11), Eq(12) and Eq(20) are all independent of the edge length of the antiprism. All the angles derived above depend on the number of sides $n$ only i.e. geometric shape of regular polygonal right antiprism.

**2.16. Construction of regular polygonal right antiprism**

A regular polygonal right antiprism can be constructed by following two methods depending on whether it is a solid or shell.

**2.16.1. Solid regular polygonal right antiprism:** The solid regular polygonal right antiprism can be made by joining all its 2n+2 elementary right pyramids, out which 2n are identical regular triangular right pyramids and 2 are identical regular polygonal right pyramids (as shown in the above Figure-1), such that all the adjacent elementary right pyramids share their mating edges, and apex points coinciding with the centre.

**2.16.2. Regular polygonal right antiprism shell:** The shell of a regular polygonal right antiprism can be made by folding about the common edges the net of all its 2n+2 faces out which 2n are identical regular triangles and 2 are identical regular polygons all having equal side. The regular polygonal faces are connected by a band of 2n equilateral triangular faces. The net of 2n+2 faces has been shown in the Figure-12 below.

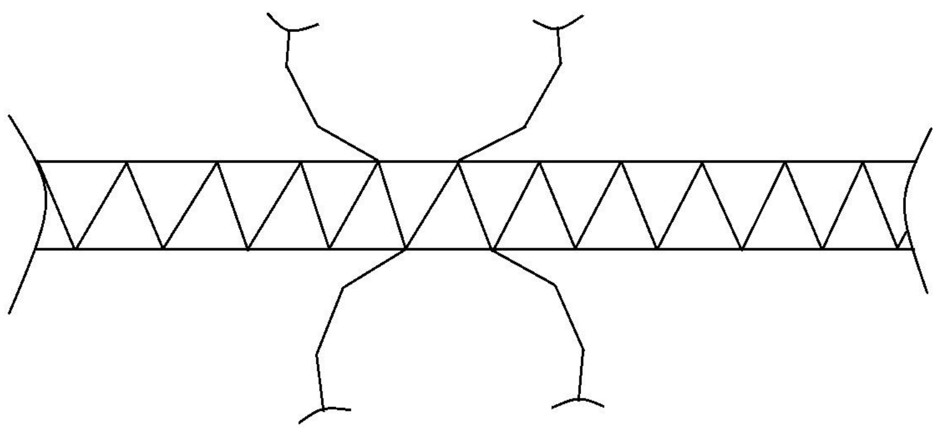

**Figure-12: The net of 2n regular triangular and 2 regular polygonal faces of a regular polygonal right antiprism.**

### 2.17. Applications of generalised formula of regular polygonal right antiprism

A regular polygonal right antiprism forms an infinite class of vertex-transitive polyhedrons. The geometric shape of regular polygonal right antiprism depends on the number of sides in polygonal base $n$ such that $n \geq 3, n \in N$. Now, substituting the different values of $n$ i.e. $n = 3, 4, 5, ....$ in the generalized formula as derived above, the infinite family of uniform n-gonal antiprism can be mathematically formulated & analysed as below.

#### 2.17.1. Regular triangular right antiprism or regular octahedron ($n = 3$)

The important geometrical parameters of a regular triangular right antiprism or octahedron (as shown in the Figure-13) can be easily determined by substituting $n = 3$ in the above generalized formula.

Number of triangular faces, $F_3 = 2n + 2 = 2 \cdot 3 + 2 = 8$

Number of edges, $E = 4n = 4 \cdot 3 = 12$

Number of vertices, $V = 2n = 2 \cdot 3 = 6$

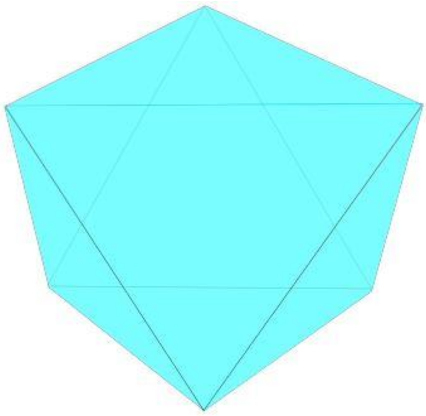

**Figure-13: Regular triangular right antiprism or octahedron (n=3).**

1) Normal distance of each equilateral triangular face from the centre of a regular triangular right antiprism or octahedron with edge length $a$ is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(7) or Eq(6) as follows

$$H_T = \frac{a}{12}\sqrt{9\mathrm{cosec}^2\frac{\pi}{2n} - 12}\Bigg|_{n=3} = \frac{a}{12}\sqrt{9\mathrm{cosec}^2\frac{\pi}{6} - 12} = \frac{a}{\sqrt{6}}$$

Or

$$H_T = H_n|_{n=3} = \frac{a}{4}\sqrt{4 - \sec^2\frac{\pi}{2n}}\Bigg|_{n=3} = \frac{a}{4}\sqrt{4 - \sec^2\frac{\pi}{6}} = \frac{a}{\sqrt{6}} \approx 0.40824829a$$

2) Perpendicular height (i.e. normal distance between opposite regular triangular faces) is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(8) as follows

$$H = 2H_n = 2 \cdot \frac{a}{4}\sqrt{4 - \sec^2\frac{\pi}{2n}}\Bigg|_{n=3} = \frac{a}{2}\sqrt{4 - \sec^2\frac{\pi}{6}} = a\sqrt{\frac{2}{3}} \approx 0.81649658a$$

3) Radius of circumscribed sphere i.e. the sphere on which all 6 identical vertices lie, is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(5) as follows

$$R_o = \frac{a}{4}\sqrt{4 + \mathrm{cosec}^2\frac{\pi}{2n}}\Bigg|_{n=3} = \frac{a}{4}\sqrt{4 + \mathrm{cosec}^2\frac{\pi}{6}} = \frac{a}{\sqrt{2}} \approx 0.707106781a$$

4) Total surface area of regular triangular right antiprism or regular octahedron is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(9) as follows

$$A_s = \frac{1}{2}na^2\left(\sqrt{3} + \cot\frac{\pi}{n}\right)\Bigg|_{n=3} = \frac{1}{2}3a^2\left(\sqrt{3} + \cot\frac{\pi}{3}\right) = 2\sqrt{3}a^2 \approx 3.464101615a^2$$

5) Volume of regular triangular right antiprism or regular octahedron is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(10) as follows

$$V = \frac{na^3}{24}\left(\cot\frac{\pi}{2n} + \cot\frac{\pi}{n}\right)\sqrt{4 - \sec^2\frac{\pi}{2n}}\Bigg|_{n=3} = \frac{3a^3}{24}\left(\cot\frac{\pi}{6} + \cot\frac{\pi}{3}\right)\sqrt{4 - \sec^2\frac{\pi}{6}} = \frac{\sqrt{2}}{3}a^3 \approx 0.47140452a^3$$

6) Dihedral angle between any two adjacent regular triangular faces having a common edge is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(21) or Eq(22) as follows

$$\theta_{TTE} = 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\text{cosec}^2\frac{\pi}{2n} - 4}\right)\Bigg|_{n=3} = 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\text{cosec}^2\frac{\pi}{6} - 4}\right)$$

$$= 2\tan^{-1}\left(\sqrt{2}\right) \approx 109.4712206^0$$

Or

$$\theta_{TTE} = \pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\Bigg|_{n=3} = \pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{6}\right) = \pi - \cos^{-1}\left(\frac{1}{3}\right) \approx 109.4712206^0$$

The above value of dihedral angle between any two triangular faces having a common edge in a regular octahedron is same as obtained by the author [6].

7) Dihedral angle between any two adjacent regular triangular faces having a common vertex is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(28) or Eq(23) as follows

$$\theta_{TTV} = \cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{n}\tan^2\frac{\pi}{2n} - \cos\frac{2\pi}{n}\right)\Bigg|_{n=3} = \cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{3}\tan^2\frac{\pi}{6} - \cos\frac{2\pi}{3}\right)$$

$$= \cos^{-1}\left(\frac{1}{3}\right) \approx 70.52877937^0$$

Or

$$\theta_{TTV} = \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\Bigg|_{n=3} = \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{6}\right) = \cos^{-1}\left(\frac{1}{3}\right) \approx 70.52877937^0$$

8) Solid angle subtended by equilateral triangular face at the centre of regular triangular right antiprism is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(11) or Eq(12) as follows

$$\omega_T = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3}{4} - \sin^2\frac{\pi}{2n}}\right)\Bigg|_{n=3} = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3}{4} - \sin^2\frac{\pi}{6}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2} \approx 1.570796327\ sr$$

Or

$$\omega_T = 2\pi - 2n\sin^{-1}\left(2\sin^2\frac{\pi}{2n}\sqrt{3 - 4\sin^2\frac{\pi}{2n}}\right)\Bigg|_{n=3} = 2\pi - 6\sin^{-1}\left(2\sin^2\frac{\pi}{6}\sqrt{3 - 4\sin^2\frac{\pi}{6}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2} \approx 1.570796327\ sr$$

9) Solid angle subtended by regular triangular right antiprism at each of its 6 identical vertices is obtained by substituting $n = 3$ in the above generalized formula i.e. Eq(20) as follows

$$\omega_V = 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3-2\cos\frac{\pi}{n}}\right)+\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right)-\sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right)-3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right)\Bigg|_{n=3}$$

$$= 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3-2\cos\frac{\pi}{3}}\right)+\sin^{-1}\left(\cos\frac{\pi}{3}\sqrt{3-2\cos\frac{\pi}{3}}\right)-\sin^{-1}\left(\frac{\cos\frac{\pi}{3}}{\cos\frac{\pi}{6}}\right)-3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{6}\right)\right)$$

$$= 8\left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) = 2\pi - 8\sin^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 1.359347638\ sr$$

The above value of solid angle subtended by a regular triangular right antiprism or octahedron at each vertex is same as obtained by the author [7].

### 2.17.2. Square right antiprism ($n = 4$)

The important geometrical parameters of a square right antiprism (as shown in the Figure-14) can be easily determined by substituting $n = 4$ in the above generalized formula.

Number of triangular faces, $F_t = 2n = 2 \cdot 4 = 8$

Number of square faces, $F_4 = 2$

Number of edges, $E = 4n = 4 \cdot 4 = 16$

Number of vertices, $V = 2n = 2 \cdot 4 = 8$

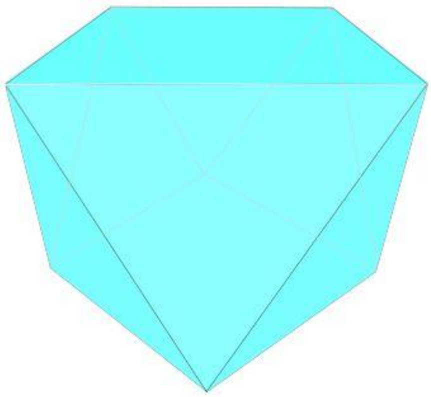

**Figure-14: Square right antiprism (n=4).**

1) Normal distance of each equilateral triangular face from the centre of a square right antiprism with edge length $a$ is obtained by substituting $n = 4$ in the above generalized formula i.e. Eq(7) as follows

$$H_T = \frac{a}{12}\sqrt{9\text{cosec}^2\frac{\pi}{2n}-12}\Bigg|_{n=4} = \frac{a}{12}\sqrt{9\text{cosec}^2\frac{\pi}{8}-12}$$

$$= \frac{a}{4}\sqrt{\frac{8+6\sqrt{2}}{3}} \approx 0.586040409a$$

2) Normal distance of each square face from the centre of square right antiprism with edge length $a$ is obtained by substituting $n = 4$ in the above generalized formula i.e. Eq(6) as follows

$$H_4 = H_n|_{n=4} = \frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}}\Bigg|_{n=4} = \frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{8}} = \frac{a}{2\sqrt{\sqrt{2}}} = \frac{a}{2^{5/4}} \approx 0.420448207a$$

3) Perpendicular height (i.e. normal distance between opposite square faces) is obtained by substituting

$n = 4$ in the above generalized formula i.e. Eq(8) as follows

$$H = 2H_n = 2 \cdot \frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}}\Bigg|_{n=4} = \frac{a}{2}\sqrt{4-\sec^2\frac{\pi}{8}} = \frac{a}{2^{1/4}} \approx 0.840896415a$$

4) Radius of circumscribed sphere i.e. the sphere on which all 8 identical vertices of square right antiprism lie, is obtained by substituting $n=4$ in the above generalized formula i.e. Eq(5) as follows

$$R_o=\frac{a}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{2n}}\Bigg|_{n=4}=\frac{a}{4}\sqrt{4+\operatorname{cosec}^2\frac{\pi}{8}}=\frac{a}{4}\sqrt{8+2\sqrt{2}}\ \approx 0.822664388a$$

5) Total surface area of square right antiprism is obtained by substituting $n=4$ in the above generalized formula i.e. Eq(9) as follows

$$A_s=\frac{1}{2}na^2\left(\sqrt{3}+\cot\frac{\pi}{n}\right)\Big|_{n=4}=\frac{1}{2}4a^2\left(\sqrt{3}+\cot\frac{\pi}{4}\right)=2a^2\left(\sqrt{3}+1\right)\approx 5.464101615a^2$$

6) Volume of square right antiprism is obtained by substituting $n=4$ in the above generalized formula i.e. Eq(10) as follows

$$V=\frac{na^3}{24}\left(\cot\frac{\pi}{2n}+\cot\frac{\pi}{n}\right)\sqrt{4-\sec^2\frac{\pi}{2n}}\Bigg|_{n=4}=\frac{4a^3}{24}\left(\cot\frac{\pi}{8}+\cot\frac{\pi}{4}\right)\sqrt{4-\sec^2\frac{\pi}{8}}$$

$$=\frac{a^3}{3}\left(2^{3/4}+2^{1/4}\right)\approx 0.956999981a^3$$

7) Dihedral angle between any two adjacent regular triangular faces having a common edge is obtained by substituting $n=4$ in the above generalized formula i.e. Eq(21) as follows

$$\theta_{TTE}=2\tan^{-1}\left(\frac{1}{2}\sqrt{3\operatorname{cosec}^2\frac{\pi}{2n}-4}\right)\Bigg|_{n=4}=2\tan^{-1}\left(\frac{1}{2}\sqrt{3\operatorname{cosec}^2\frac{\pi}{8}-4}\right)$$

$$=2\tan^{-1}\left(\frac{\sqrt{8+6\sqrt{2}}}{2}\right)\approx 127.5516029^0$$

8) Dihedral angle between any two adjacent regular triangular faces having a common vertex is obtained by substituting $n=4$ in the above generalized formula i.e. Eq(28) as follows

$$\theta_{TTV}=\cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{n}\tan^2\frac{\pi}{2n}-\cos\frac{2\pi}{n}\right)\Big|_{n=4}=\cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{4}\tan^2\frac{\pi}{8}-\cos\frac{2\pi}{4}\right)$$

$$=\cos^{-1}\left(\frac{2\sqrt{2}-3}{3}\right)\approx 93.27858947^0$$

9) Dihedral angle $\theta_{TSE}$ between any two adjacent regular triangular and square faces having a common edge is obtained by substituting $n=4$ in the above generalized formula i.e. Eq(22) as follows

$$\theta_{TSE}=\theta_{TPE}|_{n=4}=\cos^{-1}\left(-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\Bigg|_{n=4}=\cos^{-1}\left(-\frac{1}{\sqrt{3}}\tan\frac{\pi}{8}\right)$$

$$=\cos^{-1}\left(-\frac{1}{\sqrt{3}}\left(\sqrt{2}-1\right)\right)=\cos^{-1}\left(\frac{\sqrt{3}-\sqrt{6}}{3}\right)\approx 103.8361605^0$$

10) Dihedral angle $\theta_{TSV}$ between any two adjacent regular triangular and square faces having a common vertex is obtained by substituting $n=4$ in the above generalized formula i.e. Eq(23) as follows

$$\theta_{TSV}=\theta_{TPV}|_{n=4}=\cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\Bigg|_{n=4}=\cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{8}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{3}}\left(\sqrt{2}-1\right)\right) = \cos^{-1}\left(\frac{\sqrt{6}-\sqrt{3}}{3}\right) \approx 76.16383952^0$$

11) Solid angle subtended by equilateral triangular face at the centre of square right antiprism is obtained by substituting $n = 4$ in the above generalized formula i.e. Eq(11) as follows

$$\omega_T = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3}{4} - \sin^2\frac{\pi}{2n}}\right)\Bigg|_{n=4} = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3}{4} - \sin^2\frac{\pi}{8}}\right)$$

$$= 2\pi - 6\sin^{-1}\left(\frac{\sqrt{\sqrt{2}+1}}{2}\right) \approx 0.944946255\ sr$$

12) Solid angle subtended by square face at the centre of square right antiprism is obtained by substituting $n = 4$ in the above generalized formula i.e. Eq(12) as follows

$$\omega_4 = \omega_n|_{n=4} = 2\pi - 2n\sin^{-1}\left(2\sin^2\frac{\pi}{2n}\sqrt{3 - 4\sin^2\frac{\pi}{2n}}\right)\Bigg|_{n=4} = 2\pi - 8\sin^{-1}\left(2\sin^2\frac{\pi}{8}\sqrt{3 - 4\sin^2\frac{\pi}{8}}\right)$$

$$= 2\pi - 8\sin^{-1}\left(\sqrt{\frac{\sqrt{2}-1}{2}}\right) \approx 2.503400284\ sr$$

13) Solid angle subtended by square right antiprism at each of its 8 identical vertices is obtained by substituting $n = 4$ in the above generalized formula i.e. Eq(20) as follows

$$\omega_V = 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3 - 2\cos\frac{\pi}{n}}\right) + \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3 - 2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right) - 3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right)\Bigg|_{n=4}$$

$$= 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3 - 2\cos\frac{\pi}{4}}\right) + \sin^{-1}\left(\cos\frac{\pi}{4}\sqrt{3 - 2\cos\frac{\pi}{4}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{4}}{\cos\frac{\pi}{8}}\right) - 3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{8}\right)\right)$$

$$= 2\left(3\sin^{-1}\left(\frac{\sqrt{3-\sqrt{2}}}{2}\right) + \sin^{-1}\left(\sqrt{\frac{3-\sqrt{2}}{2}}\right) - \sin^{-1}\left(\sqrt{2-\sqrt{2}}\right) - 3\sin^{-1}\left(\sqrt{\frac{2-\sqrt{2}}{3}}\right)\right) \approx 1.793771333\ sr$$

### 2.17.3. Regular pentagonal right antiprism ($n = 5$)

The important geometrical parameters of a regular pentagonal right antiprism (as shown in the Figure-15 below) can be easily determined by substituting $n = 5$ in the above generalized formula.

Number of triangular faces, $F_t = 2n = 2 \cdot 5 = 10$

Number of regular pentagonal faces, $F_5 = 2$

Number of edges, $E = 4n = 4 \cdot 5 = 20$

Number of vertices, $V = 2n = 2 \cdot 5 = 10$

1) Normal distance of each equilateral triangular face from the centre of a regular pentagonal right antiprism with edge length $a$ is obtained by substituting $n = 5$ in the above generalized formula i.e. Eq(7) as follows

$$H_T = \frac{a}{12}\sqrt{9\text{cosec}^2\frac{\pi}{2n} - 12}\Bigg|_{n=5} = \frac{a}{12}\sqrt{9\text{cosec}^2\frac{\pi}{10} - 12}$$

$$= \frac{(3+\sqrt{5})a}{4\sqrt{3}} \approx 0.755761314a$$

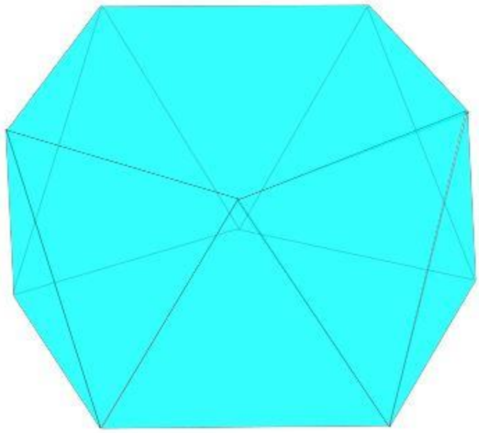

**Figure-15: Regular pentagonal right antiprism (n=5).**

2) Normal distance of each regular pentagonal face from the centre of regular pentagonal right antiprism with edge length $a$ is obtained by substituting $n = 5$ in the above generalized formula i.e. Eq(6) as follows

$$H_5 = H_n|_{n=5} = \frac{a}{4}\sqrt{4 - \sec^2\frac{\pi}{2n}}\Bigg|_{n=5} = \frac{a}{4}\sqrt{4 - \sec^2\frac{\pi}{10}}$$

$$= \frac{a}{2}\sqrt{\frac{5+\sqrt{5}}{10}} \approx 0.425325404a$$

3) Perpendicular height (i.e. normal distance between opposite regular pentagonal faces) is obtained by substituting $n = 5$ in the above generalized formula i.e. Eq(8) as follows

$$H = 2H_5 = 2\cdot\frac{a}{4}\sqrt{4 - \sec^2\frac{\pi}{2n}}\Bigg|_{n=5} = \frac{a}{2}\sqrt{4 - \sec^2\frac{\pi}{10}} = a\sqrt{\frac{5+\sqrt{5}}{10}} \approx 0.850650808a$$

4) Radius of circumscribed sphere i.e. the sphere on which all 10 identical vertices of regular pentagonal right antiprism lie, is obtained by substituting $n = 5$ in the above generalized formula i.e. Eq(5) as follows

$$R_o = \frac{a}{4}\sqrt{4 + \text{cosec}^2\frac{\pi}{2n}}\Bigg|_{n=5} = \frac{a}{4}\sqrt{4 + \text{cosec}^2\frac{\pi}{10}} = \frac{a}{4}\sqrt{10 + 2\sqrt{5}} \approx 0.951056516a$$

5) Total surface area of regular pentagonal right antiprism is obtained by substituting $n = 5$ in the above generalized formula i.e. Eq(9) as follows

$$A_s = \frac{1}{2}na^2\left(\sqrt{3} + \cot\frac{\pi}{n}\right)\Bigg|_{n=5} = \frac{1}{2}5a^2\left(\sqrt{3} + \cot\frac{\pi}{5}\right) = \frac{a^2}{2}\left(5\sqrt{3} + \sqrt{25 + 10\sqrt{5}}\right) \approx 7.77108182a^2$$

6) Volume of regular pentagonal right antiprism is obtained by substituting $n = 5$ in the above generalized formula i.e. Eq(10) as follows

$$V = \frac{na^3}{24}\left(\cot\frac{\pi}{2n} + \cot\frac{\pi}{n}\right)\sqrt{4 - \sec^2\frac{\pi}{2n}}\Bigg|_{n=5} = \frac{5a^3}{24}\left(\cot\frac{\pi}{10} + \cot\frac{\pi}{5}\right)\sqrt{4 - \sec^2\frac{\pi}{10}}$$

$$= \frac{a^3}{6}\left(5 + 2\sqrt{5}\right) \approx 1.578689326a^3$$

7) Dihedral angle between any two adjacent regular triangular faces having a common edge is obtained by substituting $n = 5$ in the above generalized formula i.e. Eq(21) as follows

$$\theta_{TTE} = 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\text{cosec}^2\frac{\pi}{2n} - 4}\right)\Bigg|_{n=5} = 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\text{cosec}^2\frac{\pi}{10} - 4}\right)$$

$$= 2\tan^{-1}\left(\frac{3+\sqrt{5}}{2}\right) \approx 138.1896851^0$$

8) Dihedral angle between any two adjacent regular triangular faces having a common vertex is obtained by substituting $n=5$ in the above generalized formula i.e. Eq(28) as follows

$$\theta_{TTV}=\cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{n}\tan^2\frac{\pi}{2n}-\cos\frac{2\pi}{n}\right)\Big|_{n=5}=\cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{5}\tan^2\frac{\pi}{10}-\cos\frac{2\pi}{5}\right)$$

$$=\cos^{-1}\left(-\frac{1}{3}\right)\approx 109.4712206^0$$

9) Dihedral angle $\theta_{TPE}$ between any two adjacent regular triangular and pentagonal faces having a common edge is obtained by substituting $n=5$ in the above generalized formula i.e. Eq(22) as follows

$$\theta_{TPE}=\theta_{TPE}|_{n=5}=\cos^{-1}\left(-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\Big|_{n=5}=\cos^{-1}\left(-\frac{1}{\sqrt{3}}\tan\frac{\pi}{10}\right)$$

$$=\cos^{-1}\left(-\sqrt{\frac{5-2\sqrt{5}}{15}}\right)=\pi-\tan^{-1}\left(3+\sqrt{5}\right)\approx 100.812317^0$$

10) Dihedral angle $\theta_{TPV}$ between any two adjacent regular triangular and pentagonal faces having a common vertex is obtained by substituting $n=5$ in the above generalized formula i.e. Eq(23) as follows

$$\theta_{TPV}=\theta_{TPV}|_{n=5}=\cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)\Big|_{n=5}=\cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{10}\right)$$

$$=\cos^{-1}\left(\sqrt{\frac{5-2\sqrt{5}}{15}}\right)=\tan^{-1}\left(3+\sqrt{5}\right)\approx 79.18768304^0$$

11) Solid angle subtended by equilateral triangular face at the centre of regular pentagonal right antiprism is obtained by substituting $n=5$ in the above generalized formula i.e. Eq(11) as follows

$$\omega_T=2\pi-6\sin^{-1}\left(\sqrt{\frac{3}{4}-\sin^2\frac{\pi}{2n}}\right)\Bigg|_{n=5}=2\pi-6\sin^{-1}\left(\sqrt{\frac{3}{4}-\sin^2\frac{\pi}{10}}\right)=\ 2\pi-6\sin^{-1}\left(\sqrt{\frac{3}{4}-\left(\frac{\sqrt{5}-1}{4}\right)^2}\right)$$

$$=2\pi-6\sin^{-1}\left(\frac{\sqrt{5}+1}{4}\right)=2\pi-6\sin^{-1}\left(\sin\frac{3\pi}{10}\right)=\frac{\pi}{5}\approx 0.62831853\ sr$$

12) Solid angle subtended by regular pentagonal face at the centre of regular pentagonal right antiprism is obtained by substituting $n=5$ in the above generalized formula i.e. Eq(12) as follows

$$\omega_5=\omega_n|_{n=5}=2\pi-2n\sin^{-1}\left(2\sin^2\frac{\pi}{2n}\sqrt{3-4\sin^2\frac{\pi}{2n}}\right)\Bigg|_{n=5}=2\pi-10\sin^{-1}\left(2\sin^2\frac{\pi}{10}\sqrt{3-4\sin^2\frac{\pi}{10}}\right)$$

$$=2\pi-10\sin^{-1}\left(\frac{\sqrt{5}-1}{4}\right)=2\pi-10\sin^{-1}\left(\sin\frac{\pi}{10}\right)=\pi\approx 3.141592654\ sr$$

13) Solid angle subtended by regular pentagonal right antiprism at each of its 10 identical vertices is obtained by substituting $n=5$ in the above generalized formula i.e. Eq(20) as follows

$$\omega_V=2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3-2\cos\frac{\pi}{n}}\right)+\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right)-\sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right)-3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right)\Bigg|_{n=5}$$

$$= 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3-2\cos\frac{\pi}{5}}\right)+\sin^{-1}\left(\cos\frac{\pi}{5}\sqrt{3-2\cos\frac{\pi}{5}}\right)-\sin^{-1}\left(\frac{\cos\frac{\pi}{5}}{\cos\frac{\pi}{10}}\right)-3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{10}\right)\right)\approx 2.059558403\ sr$$

All the above values of geometric parameters of a regular pentagonal right antiprism are same as obtained by the author [5].

Thus the analytic formula can be derived for any right antiprism having polygonal base with desired no. of sides $n$ such that $n \geq 3, n \in N$.

### 2.17.4. Infinite right antiprism ($n \to \infty$)

An infinite right antiprism is a right antiprism having two polygonal bases each with infinite number of sides i.e. $n \to \infty$ each of finite length $a$. In other words, an infinite right antiprism has a band of infinite number of equilateral triangular faces each with finite side connected by two regular polygonal bases each with infinite number of sides. Obviously, a regular polygon with infinite number of sides each of finite length looks becomes a circle. Thus an infinite right antiprism becomes a right cylinder having finite length and circular bases each with infinite radius. The important geometrical parameters of an infinite right antiprism can be easily determined by taking the limits of the above generalized formula as $n \to \infty$.

Number of triangular faces, $F_t = 2n \to \infty$

Number of regular polygonal faces with infinite no. of sides, $F_\infty = 2$

Number of edges, $E = 4n \to \infty$

Number of vertices, $V = 2n \to \infty$

1) Normal distance of each equilateral triangular face from the centre of an infinite right antiprism with finite edge length $a$ is obtained by taking limit of $H_T$ given from the above Eq(7) as $n \to \infty$

$$H_T = \lim_{n\to\infty}\frac{a}{12}\sqrt{9\text{cosec}^2\frac{\pi}{2n}-12} \to \infty$$

The above result shows that each triangular face is at an infinite distance from the centre of infinite right antiprism.

2) Normal distance of each regular polygonal face with infinite no. of sides from the centre of infinite right antiprism with finite edge length $a$ is obtained by taking limit of $H_n$ given from the above Eq(6) as $n \to \infty$

$$H_\infty = \lim_{n\to\infty} H_n = \lim_{n\to\infty}\frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}} = \frac{a\sqrt{3}}{4}$$

The above result shows that each polygonal face is at a finite distance from the centre of infinite right antiprism.

3) Perpendicular height (i.e. normal distance between opposite regular polygonal faces) is obtained by taking limit of $H$ given from the above Eq(8) as $n \to \infty$

$$H = 2H_\infty = 2\lim_{n\to\infty}\frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}} = \frac{a\sqrt{3}}{2}$$

4) Radius of circumscribed sphere i.e. the sphere on which all infinite identical vertices of infinite right antiprism lie, is obtained by taking limit of $R_o$ given from the above Eq(5) as $n \to \infty$

$$R_o = \lim_{n\to\infty}\frac{a}{4}\sqrt{4+\text{cosec}^2\frac{\pi}{2n}} \to \infty$$

5) Total surface area of infinite right antiprism is obtained by taking limit of $A_s$ given from the above Eq(9) as $n \to \infty$

$$A_s = \lim_{n\to\infty} \frac{1}{2} n a^2 \left(\sqrt{3} + \cot\frac{\pi}{n}\right) \to \infty$$

6) Volume of regular pentagonal right antiprism is obtained by taking limit of $V$ given from the above Eq(10) as $n \to \infty$

$$V = \lim_{n\to\infty} \frac{na^3}{24}\left(\cot\frac{\pi}{2n} + \cot\frac{\pi}{n}\right)\sqrt{4 - \sec^2\frac{\pi}{2n}} \to \infty$$

7) Dihedral angle between any two adjacent regular triangular faces having a common edge is obtained by taking limit of $\theta_{TTE}$ given from the above Eq(21) as $n \to \infty$

$$\theta_{TTE} = \lim_{n\to\infty} 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n} - 4}\right) = \pi$$

The above result shows that the triangular faces with finite side and common edge become co-planar with each other in infinitely long band of triangular faces.

8) Dihedral angle between any two adjacent regular triangular faces having a common vertex is obtained by taking limit of $\theta_{TTV}$ given from the above Eq(28) as $n \to \infty$

$$\theta_{TTV} = \lim_{n\to\infty} \cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{n}\tan^2\frac{\pi}{2n} - \cos\frac{2\pi}{n}\right) = \pi$$

The above result shows that the triangular faces with finite side and common vertex become co-planar with each other in infinitely long band of triangular faces.

9) Dihedral angle $\theta_{TPE}$ between any two adjacent regular triangular and polygonal faces having a common edge is obtained by taking limit of $\theta_{TPE}$ given from the above Eq(22) as $n \to \infty$

$$\theta_{TPE} = \lim_{n\to\infty} \cos^{-1}\left(-\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right) = \frac{\pi}{2}$$

The above result shows that each triangular face having common edge with regular polygonal face becomes perpendicular to the plane of polygonal base with infinite no. of sides each of finite length.

9) Dihedral angle $\theta_{TPV}$ between any two adjacent regular triangular and polygonal faces having a common vertex is obtained by taking limit of $\theta_{TPV}$ given from the above Eq(23) as $n \to \infty$

$$\theta_{TPV} = \lim_{n\to\infty} \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right) = \frac{\pi}{2}$$

The above result shows that each triangular face having common vertex with regular polygonal face becomes perpendicular to the plane of polygonal base with infinite no. of sides each of finite length.

11) Solid angle subtended by each equilateral triangular face at the centre of infinite right antiprism is obtained by taking limit of $\omega_T$ given from the above Eq(11) as $n \to \infty$

$$\omega_T = \lim_{n\to\infty} 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3}{4} - \sin^2\frac{\pi}{2n}}\right) = 0$$

The above result shows that each triangular face of finite side subtends a solid angle of 0 sr at the centre located at an infinite distance from each triangular face.

12) Solid angle subtended by each regular polygonal face with infinite no. of sides at the centre of infinite right antiprism is obtained by taking limit of $\omega_n$ given from the above Eq(12) as $n \to \infty$

$$\omega_\infty = \lim_{n\to\infty} \omega_n = \lim_{n\to\infty} 2\pi - 2n \sin^{-1}\left(2\sin^2\frac{\pi}{2n}\sqrt{3 - 4\sin^2\frac{\pi}{2n}}\right) = 2\pi$$

The above result shows that each polygonal face with infinite no. of sides subtends a solid angle of $2\pi$ sr at the centre which implies that each polygonal face covers the centre like a hemispherical cap.

13) Solid angle subtended by infinite right antiprism at each of its infinite vertices is obtained by taking limit of $\omega_V$ given from the above Eq(20) as $n \to \infty$

$$\omega_V = \lim_{n\to\infty} 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3 - 2\cos\frac{\pi}{n}}\right) + \sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3 - 2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right) - 3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right) = \pi$$

The above result shows that the infinite right antiprism subtends a solid angle of $\pi$ sr at each of its vertices which implies that the band of triangular faces becomes an infinitely long rectangular plane with finite width (as shown in the Figure-16)

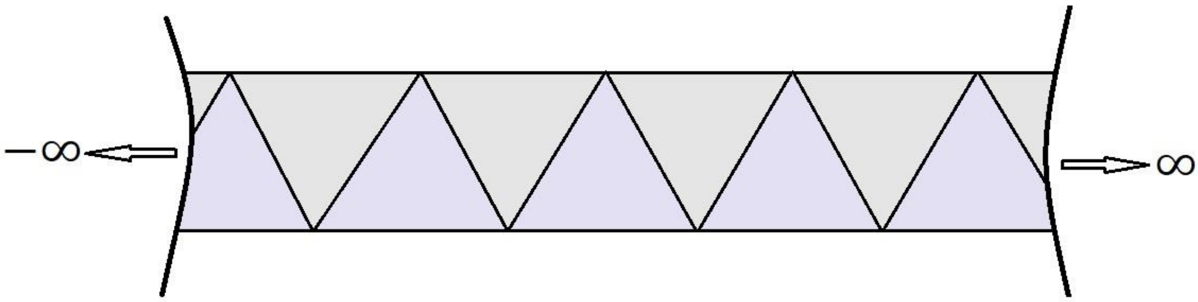


**Figure-16: The band of triangular faces of infinite right antiprism becomes a rectangular plane of infinite length & finite width, and subtends a solid angle of $\pi\ sr$ at each vertex.**

### 2.18. Variations of dimensionless parameters of regular polygonal right antiprism

The dimensionless parameters, the dihedral angles; $\theta_{TTE}, \theta_{TTV}, \theta_{TPE}$ & $\theta_{TPV}$ and the solid angles; $\omega_T, \omega_n$ & $\omega_V$ (as derived above) only depend on the number of sides in regular polygonal base $n$ which is a natural number such that $n \geq 3, n \in N$. The graphs of variation of dihedral and solid angles with respect to the number of sides $n$ can be plotted by assuming $n$ to be a continuous variable such that $n \geq 3$ as shown by the solid curves in the Figure-17 and Figure-18 below. These plots can be used to determine the values of dihedral and solid angles of a regular n-gonal right antiprism at the desired positive integer value of n.

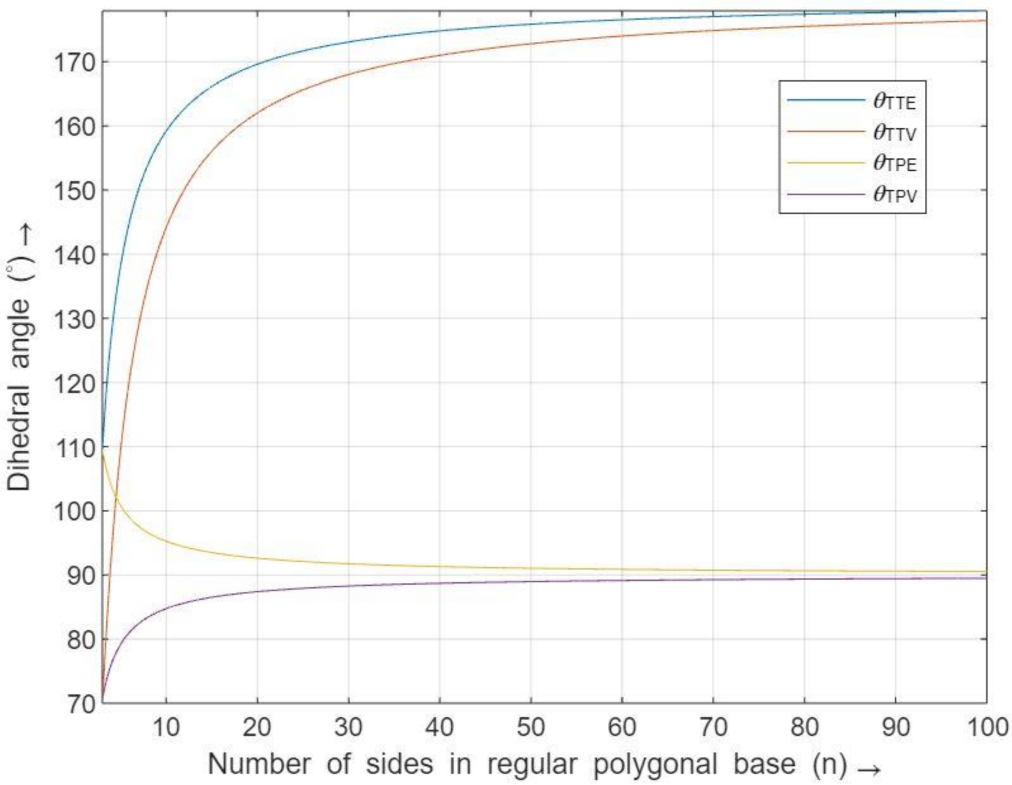


**Figure-17: The variations of dihedral angles $\theta_{TTE}, \theta_{TTV}, \theta_{TPE}$ and $\theta_{TPV}$ w.r.t. n**

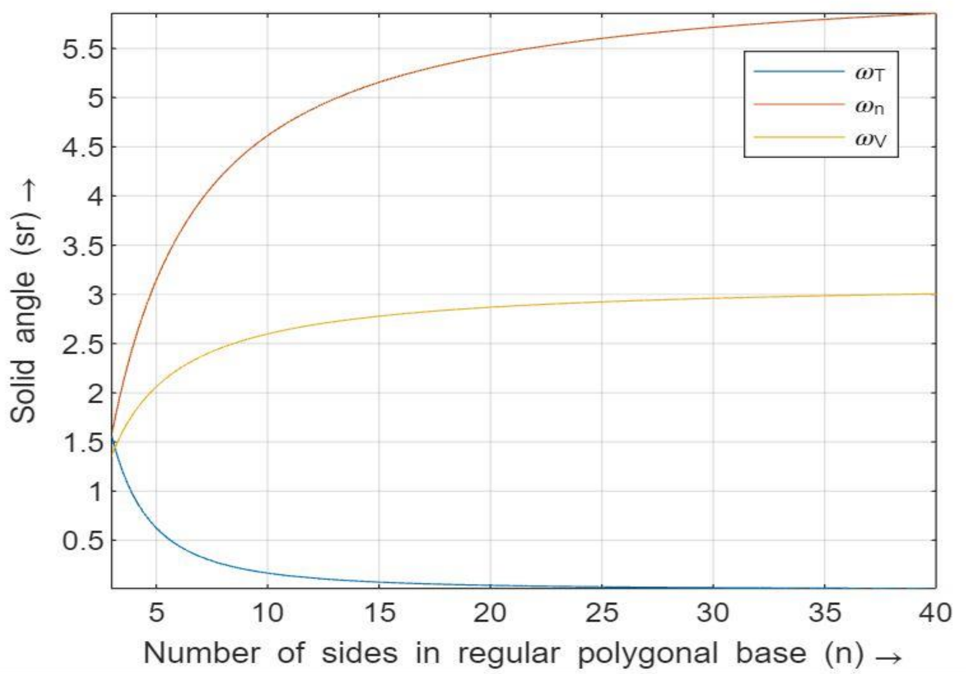


**Figure-18: The variations of solid angles $\omega_T, \omega_n$ and $\omega_V$ w.r.t. n**

**Summary:** All the important geometric parameters of a regular n-gonal right antiprism having edge length $a$ can be determined as tabulated below.

| Geometric parameter | Formula |
| --- | --- |
| Normal distance of equilateral triangular face from the centre | $H_T = \frac{a}{12}\sqrt{9\mathrm{cosec}^2\frac{\pi}{2n} - 12} = \frac{a}{4\sqrt{3}}\cot\frac{\pi}{2n}\sqrt{4-\sec^2\frac{\pi}{2n}}$ |
| Normal distance of regular polygonal face from the centre | $H_n = \frac{a}{4}\sqrt{4-\sec^2\frac{\pi}{2n}}$ |
| Perpendicular height (i.e. normal distance between opposite regular polygonal faces) | $H = 2H_n = \frac{a}{2}\sqrt{4-\sec^2\frac{\pi}{2n}}$ |
| Radius of circumscribed sphere | $R_o = \frac{a}{4}\sqrt{4+\mathrm{cosec}^2\frac{\pi}{2n}}$ |
| Total surface area | $A_s = \frac{1}{2}na^2\left(\sqrt{3}+\cot\frac{\pi}{n}\right)$ |
| Volume | $V = \frac{na^3}{24}\left(\cot\frac{\pi}{2n}+\cot\frac{\pi}{n}\right)\sqrt{4-\sec^2\frac{\pi}{2n}}$ |
| Dihedral angle between any two adjacent regular triangular faces having a common edge | $\theta_{TTE} = 2\tan^{-1}\left(\frac{1}{2}\sqrt{3\mathrm{cosec}^2\frac{\pi}{2n}-4}\right)$ |
| Dihedral angle between any two adjacent regular triangular faces having a common vertex | $\theta_{TTV} = \cos^{-1}\left(-\frac{2}{3}\sin^2\frac{\pi}{n}\tan^2\frac{\pi}{2n}-\cos\frac{2\pi}{n}\right)$ |
| Dihedral angle between adjacent regular triangular and polygonal faces having a common edge | $\theta_{TPE} = \pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)$ |
| Dihedral angle between adjacent regular triangular and polygonal faces having a common vertex | $\theta_{TPV} = \pi - \theta_{TPE} = \cos^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\pi}{2n}\right)$ |
| Solid angle subtended by equilateral triangular face at the centre | $\omega_T = 2\pi - 6\sin^{-1}\left(\sqrt{\frac{3}{4}-\sin^2\frac{\pi}{2n}}\right)$ |
| Solid angle subtended by regular polygonal face at the centre | $\omega_n = 2\pi - 2n\sin^{-1}\left(2\sin^2\frac{\pi}{2n}\sqrt{3-4\sin^2\frac{\pi}{2n}}\right)$ |
| Solid angle subtended by polygonal antiprism at each of its 2n identical vertices lying on a sphere | $\omega_V = 2\left(3\sin^{-1}\left(\frac{1}{2}\sqrt{3-2\cos\frac{\pi}{n}}\right)+\sin^{-1}\left(\cos\frac{\pi}{n}\sqrt{3-2\cos\frac{\pi}{n}}\right) - \sin^{-1}\left(\frac{\cos\frac{\pi}{n}}{\cos\frac{\pi}{2n}}\right) - 3\sin^{-1}\left(\frac{2}{\sqrt{3}}\sin\frac{\pi}{2n}\right)\right)$ |

## Conclusions

In this paper, the generalized formulas have been derived in terms of edge length and number of sides of regular polygonal base of the regular polygonal right antiprism for computing its important parameters such as, normal distances of faces from the centre, normal height, radius of circumscribed sphere, surface area, volume, dihedral angles between adjacent faces, solid angles subtended by the faces at the centre and solid angle subtended by antiprism at each vertex. The analytic and generalized formula derived here can be used to mathematically analyse and formulate the polyhedrons with large no. of faces, edges and vertices in discrete geometry.

## References

[1]: Weisstein, Eric W. "Antiprism." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/Antiprism.html

[2]: Wikipedia. "Antiprism". https://en.wikipedia.org/wiki/Antiprism . Accessed on 16-04-2023.

[3]: Rajpoot, Harish Chandra. (2019). "HCR's Theory of Polygon" "Solid angle subtended by any polygonal plane at any point in the space".

Available at:

https://www.researchgate.net/publication/333641300_HCR's_Theory_of_Polygon_Solid_angle_subtended_by_any_polygonal_plane_at_any_point_in_the_space

[4]: Rajpoot, Harish Chandra (2014). HCR's Inverse Cosine Formula (Solution of internal & external angles of a tetrahedron). Academia.edu

[5]: Rajpoot, Harish Chandra. (2023). Mathematical analysis of regular pentagonal right antiprism. 10.13140/RG.2.2.22879.33444.

[6]: Rajpoot, Harish Chandra. (2015). Tables for dihedral angles between the adjacent faces with a common edge of various uniform polyhedral.

Available at:

https://www.academia.edu/11590798/Tables_for_dihedral_angles_between_the_adjacent_faces_with_a_common_edge_of_various_regular_and_uniform_polyhedra_by_HCR

[7]: Rajpoot, Harish Chandra. (2015). Solid angles subtended by the platonic solids (regular polyhedra) at their vertices.

Available at:

https://www.academia.edu/11711978/Solid_angles_subtended_by_the_platonic_solids_regular_polyhedrons_at_their_vertices_by_HCR